HW3 PHYS168 lasers Spring 2012 Prepared by N. Eradat Due date Feb 27, 2012

10 each

- V1.4, 1.7, 1.8, 1.10, 2.1, 2.3, 2,5, 2.6.
- Problems from review

1.16 1 a) Prove that for a harmonic plane electromagnetic wave $\mathbf{E}=\mathbf{E}_0\cos(\mathbf{k}\cdot\mathbf{r}-\omega t)$, traveling through an insulating isotropic medium, $|\mathbf{E}|=V|\mathbf{B}|$ and in vacuum $|\mathbf{E}|=c|\mathbf{B}|$.

b) Calculate the magnitude of the poynting vector

c) Prove that the energy content of this plane wave, in unit volume, due to electric and magnetic fields is equal.

d) Calculate the total energy per unit volume.

1.17 2 a) Calculate time average of the harmonic plane wave $E=E_0e^{i(kx-\omega t)}$ over a time of T (T is not temporal period of the wave).

b) Calculate the $\langle\cos(kx-\omega t)\rangle_T$, $\langle\sin(kx-\omega t)\rangle_T$, $\langle\cos(kx-\omega t)\rangle_T^2$, $\langle\sin(kx-\omega t)\rangle_T^2$. Use the notation $\operatorname{sin}c\ x=\dfrac{\sin x}{x}$.

c) Use the results of parts a and b to show that for plane waves in vacuum $I\equiv\langle\mathbf{S}\rangle_T=\dfrac{c\varepsilon_0}{2}E_0^2$

d) Calculate the optical flux density for the plane EM wave with the following electrical (also called optical) field moving in vacuum

$$E_x=E_y=0,\ E_z=100\sin\left[8\pi\times10^{14}\left(t-\frac{x}{3\times10^8}\right)\right].\ (Ans:\ 13.3\ W/m^3)$$

2.a $\langle f(t) \rangle_T = \frac{1}{T}\int_{t-T/2}^{t+T/2} f(t)\, e^{-i(kx-\omega t)}\, dt$

First let's tak time average of $e^{i\omega t}$ with that in mind T is not the period

$$\langle e^{i\omega t} \rangle_T = \frac{1}{T}\int_{t-T/2}^{t+T/2} e^{i\omega t}\, dt = \frac{1}{i\omega T}\left[e^{i\omega t}\right]_{t-T/2}^{t+T/2}$$

$$\langle e^{i\omega t} \rangle_T = \frac{1}{\omega T/2}\left[\left(e^{i\omega t}\left(\frac{e^{i\omega T/2} - e^{-i\omega T/2}}{2i}\right)\right)\right.$$

$$= e^{i\omega t}\frac{\sin(\omega T/2)}{\omega T/2} = \boxed{e^{i\omega t}\,\mathrm{sinc}\left(\frac{\omega T}{2}\right) = \langle e^{i\omega T} \rangle_P}$$

$$\boxed{\langle e^{i\omega t} \rangle_T = \mathrm{Sinc}\left(\frac{\omega T}{2}\right) e^{i\omega t}}$$

Then $\langle \cos\omega t \rangle_T = \frac{1}{T}\int \cos\omega t\, dt = \frac{1}{\omega T}\sin\omega t\Big|_{t-T/2}^{t+T/2}$

$$\langle \cos\omega t \rangle_T = \frac{1}{\omega T}\sin\left(\omega t + \omega\frac{T}{2}\right) - \sin\left(\omega t - \omega T/2\right)$$

$$= \frac{1}{\omega T}\left[\sin\omega t \cancel{\cos\frac{\omega T}{2}} + \cos\omega t \sin\frac{\omega T}{2} - \sin\omega T \cancel{\cos\frac{\omega T}{2}} + \cos\omega t \sin\frac{\omega T}{2}\right]$$

$$= \frac{1}{\omega T}\, 2\cos\omega t \sin\omega\frac{T}{2} = \cos\omega t\, \frac{\sin \omega T/2}{\omega T/2}$$

$\boxed{\langle \cos\omega t \rangle_T = \cos\omega t\, \mathrm{Sinc}\left(\frac{\omega T}{2}\right)}$ For optical frequencies

$\boxed{\langle \sin\omega t \rangle_T = \sin\omega t\, \mathrm{Sinc}\left(\frac{\omega T}{2}\right)}$ 10^{14} Hz even milisec. averaging leads to zero since sinc becomes

zero very rapidly & $\boxed{\langle \cos\omega t \rangle_T = 0}$ $\boxed{\langle \sin\omega t \rangle_T = 0}$

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Phys 468

1 $\left\{ \begin{array}{l} \vec{P} = \vec{S} = \frac{1}{\mu} \vec{E} \times \vec{B} = \vec{E} \times \vec{H} \\ w_e = \frac{1}{2} \epsilon E^2 = \frac{1}{2} \epsilon E E^* \quad \text{unknown} \\ w_m = \frac{1}{2} \mu H^2 = \frac{1}{2\mu} B \cdot B^* \end{array} \right.$

Assume E is in y direction

$P_\sigma = \sigma E^2 = 0$ in this case

- Isotropic medium: direction of energy flow is the same as direction of propagation.
- Insulating: $\sigma = 0$ or $\vec{J} = 0$
- EM wave: so it is transvese $E \perp B \perp k$

1.a) a) $\vec{E} = E_y \cos(kx - wt)\hat{j} \quad \vec{B} = \quad \cos(kx - wt)\hat{k}$

Using Maxwell equation $\nabla \times E = -\frac{\partial B}{\partial t}$ we can see

$\nabla \times E_y \cos(kx - wt)\hat{i} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & E & 0 \end{vmatrix} = + \frac{\partial E}{\partial z} i + \frac{\partial E}{\partial x} \hat{k}$

$\nabla \times \vec{E} = -E_y k \sin(kx - wt)\hat{k}$

$B = -\int \nabla \times E \, dt = -\int -E_y k \sin(kx - wt) \hat{k} \, dt$

$\vec{B} = +E_y k \frac{1}{w} \cos(kx - wt)\hat{k} = \boxed{\frac{1}{V} E_y \cos(kx - wt)\hat{k} = \vec{B}}$

$|\vec{E}| = V|B|$ for EM waves $V = c \quad |E| = c|B|$

1.b) b) Pointing vector: $|\vec{P}| = \left|\frac{1}{\mu} E \times B\right| = \frac{1}{\mu} \begin{vmatrix} i & j & k \\ 0 & \vec{E}_j & 0 \\ 0 & 0 & B_k \end{vmatrix} = \left|\frac{1}{\mu}(E_j B_k)\hat{i}\right|$

$\vec{P} = \frac{1}{\mu} E_y \cos(kx - wt) \frac{1}{V} E_y (\cos kx - wt)\hat{i}$

$\vec{P} = \frac{1}{\mu} \sqrt{\mu\epsilon} E_y^2 \cos^2(kx - wt)\hat{i} \Rightarrow \boxed{|\vec{P}| = \sqrt{\frac{\epsilon}{\mu}} E_y^2 \cos^2(kx - wt)}$

1.c) c) $w_e = \frac{1}{2} \epsilon E^2 = \frac{1}{2} \epsilon E_y^2 \cos^2(kx - wt)$

$w_m = \frac{1}{2\mu} B^2 = \frac{1}{2\mu} \mu\epsilon E_y^2 \cos^2(kx - wt) = \frac{1}{2} \epsilon E^2$

So $w_e = w_m$ for plane waves

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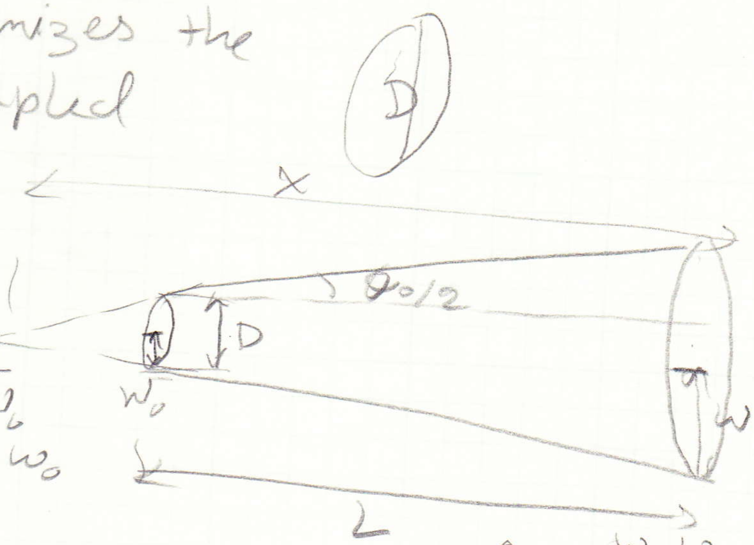
1.d) d) $u = u_e + u_m = \frac{1}{2}\epsilon E^2 + \frac{1}{2}\epsilon E^2 = \epsilon E^2$

1.4 Find D that minimizes the Volume of gas sampled

$\lambda = 632.8\ nm$

$L = 10\ m$



$\theta_0 = \frac{2\lambda}{\pi w_0} \Rightarrow \frac{\theta_0}{2} = \frac{\lambda}{\pi w_0}$

we assume $D = 2w_0$

$$V = \frac{1}{3}\pi w^2 x - \frac{1}{3}\pi w_0^2 (x-L) \qquad \frac{\theta}{2} = \frac{w-w_0}{L}$$

$$\tan\frac{\theta}{2} = \frac{w}{x} = \frac{w_0}{x-L} \Rightarrow \begin{cases} x = \frac{w}{\tan\frac{\theta}{2}} = \frac{Lw}{w-w_0} \\ x-L = \frac{w_0}{\tan\frac{\theta}{2}} = \frac{Lw_0}{w-w_0} \end{cases}$$

$$V = \frac{\pi}{3}\left[w^2\frac{Lw}{w-w_0} - w_0^2\frac{Lw_0}{w-w_0}\right]$$

$$V = \frac{\pi L}{3}\left[\frac{w^3-w_0^3}{w-w_0}\right] = \frac{\pi L}{3}\left[\frac{\cancel{(w-w_0)}(w^2+ww_0+w_0^2)}{\cancel{(w-w_0)}}\right]$$

$$V = \frac{\pi L}{3}\left[w^2+ww_0+w_0^2\right]$$

$$\frac{w_0}{L} + \frac{\theta}{2} = \frac{w}{L} \Rightarrow w = L\left(\frac{w_0}{L}+\frac{\theta_0}{2}\right) = w_0 + L\frac{\lambda}{\pi w_0}$$

$$V = \frac{\pi L}{3}\left[\left(w_0+\frac{L\lambda}{\pi w_0}\right)^2 + \left(w_0+\frac{L\lambda}{\pi w_0}\right)w_0 + w_0^2\right]$$

$$= \frac{\pi L}{3}\left[w_0^2 + \frac{L^2\lambda^2}{\pi^2 w_0^2} + \frac{2L\lambda}{\pi} + w_0^2 + \frac{L\lambda}{\pi} + w_0^2\right] = \frac{\pi L}{3}\left[3w_0^2 + \left(\frac{L\lambda}{\pi}\right)^2\frac{1}{w_0^2} + \frac{3L\lambda}{\pi}\right]$$

$$\frac{dV}{dw_0} = 6w_0 - \left(\frac{L\lambda}{\pi}\right)^2\frac{2}{w_0^3} = 0 \Rightarrow w_0^4 = \frac{1}{3}\left(\frac{L\lambda}{\pi}\right)^2 = \left(\frac{D}{2}\right)^4$$

$$\boxed{D^4 = \frac{4}{3}\left[\frac{2L\lambda}{\pi}\right]^2}$$ the diameter that would minimize the volume.

1-4 Continued

$$D = \left[\frac{16}{3}\frac{\lambda^2 L^2}{\pi^2}\right]^{1/4} = \left[\frac{16}{3}\frac{(632.8\times10^{-9}m)^2(10m)^2}{(3.14)^2}\right]^{1/4}$$

$$\boxed{D = 2.16\times10^{-3}m}$$

$$\frac{\theta}{2} = \frac{2\lambda}{\pi D} = \frac{2(632.8\times10^{-9}m)}{(3.14)(2.16\times10^{-3}m)} \Rightarrow \boxed{\frac{\theta}{2} = 1.87\times10^{-4}\,rad}$$

$$W = W_0 + \frac{L\theta}{2} = \frac{D}{2} + \frac{L\theta}{2} = \frac{2.16\times10^{-3}m}{2} + (10m)(1.87\times10^{-4}\,rad)$$

$$\boxed{W = 2.95\times10^{-3}m}$$

1.7 $\mathcal{F}\{E(y)\} = E(k_y)$

$$E(y) = E_0 e^{-\left(\frac{y}{w_0}\right)^2}$$

$$E(k_y) = \pi^{1/2} w_0 E_0 e^{-\left(\frac{k_y w_0}{2}\right)^2}$$

$$E(k_y) = E_0 \int_{-\infty}^{\infty} e^{-\left(\frac{y}{w_0}\right)^2} e^{-ik_y y}\, dy$$

$$\left(\frac{y}{w_0}\right)^2 + ik_y y + \left(\frac{i k_y w_0}{2}\right)^2 - \left(\frac{i k_y w_0}{2}\right)^2$$

$$\left(\frac{y}{w_0} + i\frac{k_y w_0}{2}\right)^2 + \frac{k_y^2 w_0^2}{4} = \left(\frac{y}{w_0}\right)^2 + ik_y y$$

$$E(k_y) = E_0 w_0 e^{-\frac{k_y^2 w_0^2}{4}} \underbrace{\int_{-\infty}^{\infty} e^{-u^2}\, du}_{\sqrt{\pi}}$$

$$\boxed{E(k_y) = w_0 E_0 \sqrt{\pi}\, e^{-\frac{k_y^2 w_0^2}{4}}}$$

1.8 $\Delta x \Delta k_x = \frac{1}{2}$

$$\Delta x^2 = \frac{\int x^2 |E(x)|^2 dx}{\int |E(x)|^2 dx} \qquad \Delta k_x^2 = \frac{\int k_x^2 |E(k_x)| dk_x}{\int |E(k_x)|^2 dx}$$

For $E(x) = E_0 e^{-\left(\frac{x}{w_0}\right)^2}$ what are Δx & Δk_x

$$\Delta x^2 = \frac{\int x^2 E_0^2 e^{-2\left(\frac{x}{w_0}\right)^2} dx}{\int E_0^2 e^{-2\left(\frac{x}{w_0}\right)^2} dx} = \frac{\int \frac{E_0^2 w_0^3}{2\sqrt{2}} u^2 e^{-u^2} du}{\int E_0^2 \frac{w_0}{\sqrt{2}} e^{-u^2} du} \quad u = \frac{\sqrt{2}x}{w_0} \quad du = \frac{\sqrt{2}}{w_0} dx \quad x^2 = \frac{w_0^2}{2} u^2$$

$$\Delta x^2 = \frac{w_0^2}{2} \frac{\int_{-\infty}^{\infty} u^2 e^{-u^2} du}{\int_{-\infty}^{\infty} e^{-u^2} du} \quad \text{with} \int_0^{\infty} x^m e^{-ax^2} dx = \frac{\Gamma(m+1)/2}{2a^{(m+1)/2}} \quad \Gamma(\tfrac{1}{2}) = \sqrt{\pi}$$

$$\Gamma(\tfrac{3}{2}) = \frac{\sqrt{\pi}}{2}$$

$$\Delta x^2 = \frac{w_0^2}{2} \frac{\Gamma(\frac{3}{2})}{\Gamma(\frac{1}{2})} = \frac{w_0^2}{2} \frac{\sqrt{\pi}}{2\sqrt{\pi}} \Rightarrow \boxed{\Delta x^2 = \frac{w_0^2}{4}} \qquad \Gamma(\tfrac{5}{2}) = 3\sqrt{\tfrac{\pi}{4}}$$

Same method $\Delta k_x^2 = \frac{1}{w_0^2}$ & $\boxed{\Delta x^2 \Delta k_x^2 = \frac{1}{4}}$

1.8 continued

For $TEM_{1,0}$

$$\Delta x^2 = \frac{3}{2}\frac{w_0^2}{2}$$

$$\Delta k_x^2 = \frac{2}{w_0^2}\frac{\Gamma(5/2)}{\Gamma(3/2)} = \frac{2}{w_0^2}\frac{3\sqrt{\pi}/4}{\sqrt{\pi}/2}$$

$$\Delta x \Delta k_x = \frac{\sqrt{3}\,w_0}{2}\frac{\sqrt{3}}{w_0} \Rightarrow \boxed{\Delta x \Delta k_x = \frac{3}{2}}$$

1.10

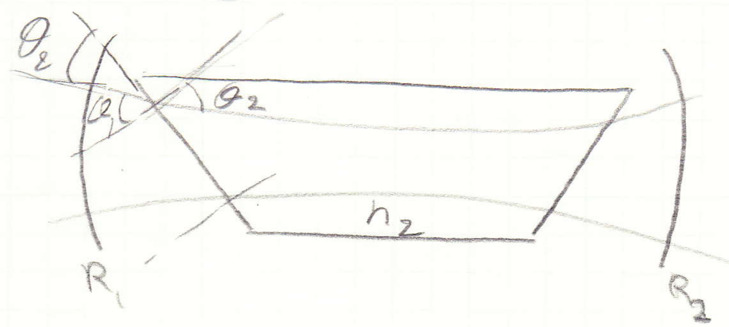
$n[\text{quartz}] = 1.43$

θ from the axis of cavity

$$\tan\theta_1 = \frac{n_2}{n_1} = 1.43$$

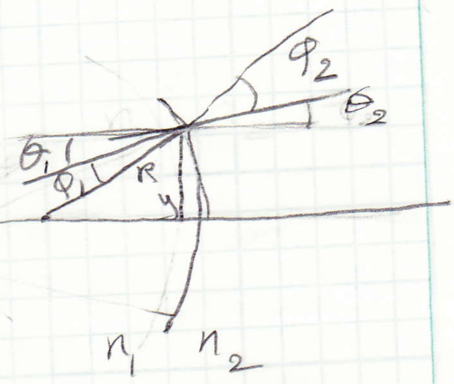
$$\theta_1 = 55^\circ$$

$$\theta_2 = 90 - \theta_1 = 35$$



HW3

2.1 r_1 & r_2 beam heights $r_1 = r_2$



r_1' & r_2' be slopes

$$\begin{pmatrix} r_2 \\ r_2' \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} r_1 \\ r_1' \end{pmatrix}$$

$$r_1 = r_2 \Rightarrow r_2 = A r_1 + B r_1' \Rightarrow \boxed{B = 0} \quad \boxed{A = 1}$$

$$\varphi_1 + \theta_1 = \varphi_2 + \theta_2 \Rightarrow \theta_2 = \varphi_1 + \theta_1 - \varphi_2$$

$$r_2' = \varphi_1 + r_1' - \varphi_2 \Rightarrow r_2' = r_1' + \varphi_1\left(1 - \frac{n_1}{n_2}\right)$$

$$n_1 \varphi_1 = n_2 \varphi_2$$

$$\sin(\theta_1 + \varphi_1) = \frac{r_1}{R} \approx \theta_1 + \varphi_1 = r_1' + \varphi_1 \Rightarrow \varphi_1 = \frac{r_1}{R} - r_1'$$

$$r_2' = r_1' + \left(\frac{r_1}{R} - r_1'\right)\left(1 - \frac{n_1}{n_2}\right)$$

$$r_2' = \frac{1}{R}\left(1 - \frac{n_1}{n_2}\right) r_1 + \left(1 - 1 + \frac{n_1}{n_2}\right) r_1'$$

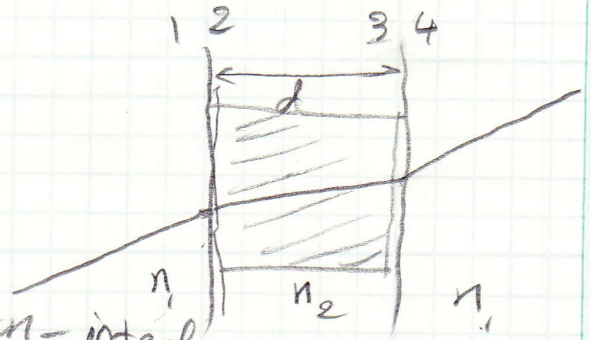
$$r_2' = C r_1 + D r_1'$$

$$\boxed{C = \frac{1}{R}\left(1 - \frac{n_1}{n_2}\right)} \quad \boxed{D = \frac{n_1}{n_2}}$$

The RTM is $\begin{bmatrix} 1 & 0 \\ \frac{1}{R}\left(1 - \frac{n_1}{n_2}\right) & \frac{n_1}{n_2} \end{bmatrix}$

2.3 RTM for a plane dielectric slab of length $\underline{d}$

RTM of an interface



$$\begin{bmatrix} 1 & 0 \\ 0 & \frac{n_1}{n_2} \end{bmatrix}$$

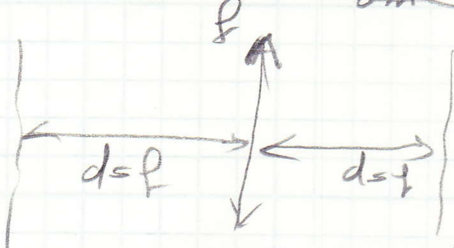
We apply interface-propagation-interface

$$\begin{bmatrix} r_f \\ r_f' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{n_2}{n_1} \end{bmatrix} \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{n_1}{n_2} \end{bmatrix} \begin{bmatrix} r_1 \\ r_1' \end{bmatrix}$$

$$\boxed{RTM = \begin{bmatrix} 1 & \frac{n_1}{n_2} d \\ 0 & 1 \end{bmatrix}}$$

$$AD - BC = \frac{n_{in}}{n_{out}} = \frac{1}{1} = \underline{1}$$

2.5



$$T = \begin{bmatrix} 1 & 0 \\ 0 & n_0 \end{bmatrix} \begin{bmatrix} \cos\frac{d}{\ell} & \ell \sin\frac{d}{\ell} \\ -\frac{1}{\ell}\sin\frac{d}{\ell} & \cos\frac{d}{\ell} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{n_0} \end{bmatrix}$$

A) $$T = \begin{bmatrix} 1 & 0 \\ 0 & n_0 \end{bmatrix} \begin{bmatrix} \cos\frac{d}{\ell} & \frac{\ell}{n_0}\sin\frac{d}{\ell} \\ -\frac{1}{\ell}\sin\frac{d}{\ell} & \frac{1}{n_0}\cos\frac{d}{\ell} \end{bmatrix} \Rightarrow \boxed{T = \begin{bmatrix} \cos\frac{d}{\ell} & \frac{\ell}{n_0}\sin\frac{d}{\ell} \\ -\frac{n_0}{\ell}\sin\frac{d}{\ell} & \cos\frac{d}{\ell} \end{bmatrix}}$$

B) $$d = \frac{\pi \ell}{2} \Rightarrow \boxed{T = \begin{bmatrix} 0 & \frac{\ell}{n_0} \\ -\frac{n_0}{\ell} & 0 \end{bmatrix}}$$

$$AD - BC = +1$$

C) $$T = \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{bmatrix} \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & d \\ -\frac{1}{f} & 1-\frac{d}{f} \end{bmatrix} = \begin{bmatrix} 1-\frac{d}{f} & d+d(1-\frac{d}{f}) \\ -\frac{1}{f} & 1-\frac{d}{f} \end{bmatrix}$$

For $d=f$ $$\boxed{T = \begin{bmatrix} 0 & f \\ -1/f & 0 \end{bmatrix}}$$

2.5 d

$n(r) = n_0\left[1 - \frac{1}{2}\left(\frac{r}{\ell}\right)^2\right]$

$n_0 = 1.53$

$n(a) = 1.525$

$a = 2\text{ mm}$

$f = ?$

$f = \frac{\ell}{n_0}$

We need with ℓ

$1 - \frac{n(r)}{n_0} = \frac{1}{2}\left(\frac{r}{\ell}\right)^2$

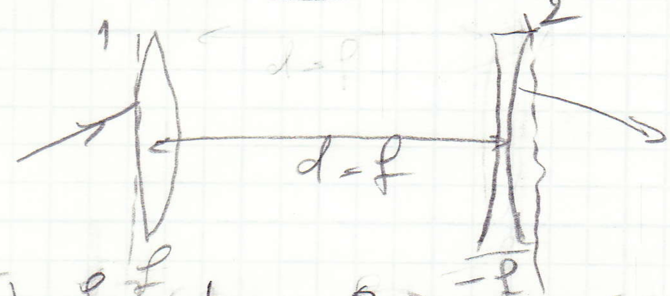
$\frac{r}{\ell} = 2\left(1 - \frac{n(r)}{n_0}\right)^{1/2}$

$\ell = \frac{r}{2\sqrt{1 - \frac{n(r)}{n_0}}} = \frac{2\text{ mm}}{2\left(1 - \frac{1.525}{1.53}\right)^{1/2}}$

$\ell = 17.49\text{ mm}$

$f = \frac{17.49\text{ mm}}{1.53} \Rightarrow \boxed{f = 11.43\text{ mm}}$ equivalent convex lens

2.6 ABCD Matrix



$$T = \begin{bmatrix} 1 & 0 \\ +\frac{1}{f} & 1 \end{bmatrix}\begin{bmatrix} 1 & f \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{bmatrix}\begin{bmatrix} 0 & f \\ -\frac{1}{f} & 1 \end{bmatrix}$$

$$\boxed{T = \begin{bmatrix} 0 & f \\ -\frac{1}{f} & 2 \end{bmatrix}}$$