Superposition of waves (review)

PHYS 168

Lasers

SJSU Spring 2012 Eradat

Superposition of waves

- Superposition of waves is the common conceptual basis for some optical phenomena such as:
  - Polarization
  - Interference
  - Diffraction
- What happens when two or more waves overlap in some region of space.
- How the specific properties of each wave affects the ultimate form of the composite disturbance?
- Can we recover the <u>ingredients</u> of a complex disturbance?

Linearity and superposition principle

The scaler 3D wave equation $\frac{\partial^2 \psi(r,t)}{\partial r^2} = \frac{1}{V^2}\frac{\partial^2 \psi(r,t)}{\partial t^2}$ is a linear differential equation (all derivatives apper in first power). So any linear combination of its solutions $\psi(r,t) = \sum_{i=1}^{n} C_i \psi_i(r,t)$ is a solution.

Superposition principle: resultant disturbance at any point in a medium is the algebraic sum of the separate constituent waves.

We focus only on linear systems and scalar functions for now.

At high intensity limits most systems are nonlinear.

Example: Inensity of a typical focused laser beam $=\sim 10^{10}\ W/cm^2$ compared to sun light on earth $\sim 10\ W/cm^2$.

Electric field of the laser beam can trigger nonlinear phenomena.

Superposition of two waves

Two light rays with same frequency meet at point p traveled by x_1 and x_2

$$E_1 = E_{01}\sin\left[\omega t - (kx_1 + \varepsilon_1)\right] = E_{01}\sin\left[\omega t + \alpha_1\right]$$

$$E_2 = E_{02}\sin\left[\omega t - (kx_2 + \varepsilon_2)\right] = E_{02}\sin\left[\omega t + \alpha_2\right]$$

Where $\alpha_1 = -(kx_1 + \varepsilon_1)$ and $\alpha_2 = -(kx_2 + \varepsilon_2)$

Magnitude of the composite wave is sum of the magnitudes at a point in space & time or: $E = E_1 + E_2 = E_0 \sin(\omega t + \alpha)$ where

$$E_0^2 = E_{01}^2 + E_{02}^2 + \underline{2E_{01}E_{02}\cos(\alpha_2 - \alpha_1)} \text{ and } \tan\alpha = \frac{E_{01}\sin\alpha_1 + E_{02}\sin\alpha_2}{E_{01}\cos\alpha_1 + E_{02}\cos\alpha_2}$$

The resulting wave has <u>same frequency</u> but <u>different amplitude and phase</u>.

$2E_{01}E_{02}\cos(\alpha_2 - \alpha_1)$ is the <u>interference term</u>

$\delta \equiv \alpha_2 - \alpha_1$ is the <u>phase difference</u>.

Phase difference and interference

$$\delta \equiv \alpha_2 - \alpha_1 = (kx_1 + \varepsilon_1) - (kx_2 + \varepsilon_2) = \frac{2\pi}{\lambda}(x_1 - x_2) + (\varepsilon_1 - \varepsilon_2) = \delta_1 + \delta_2$$

Total phase difference between the two waves has two different origins.

a) $\delta_2 = (\varepsilon_1 - \varepsilon_2)$ phase difference due to the initial phase of the waves.

Waves with constant initial phase difference are said to be coherent.

b) $\delta_1 = \frac{2\pi}{\lambda_0} n(x_1 - x_2) = k_0 \Lambda$ is pahse difference due to the Optical Path

Difference or OPD $\equiv \boxed{\Lambda = n(x_1 - x_2)}$

Waves in-phase: $\delta \equiv \alpha_2 - \alpha_1 = 0, \pm 2\pi, \pm 4\pi, \ldots$ then E_0 is mximum

Waves out of phase: $\delta \equiv \alpha_2 - \alpha_1 = \pm \pi, \pm 3\pi, \ldots$ then E_0 is minimum

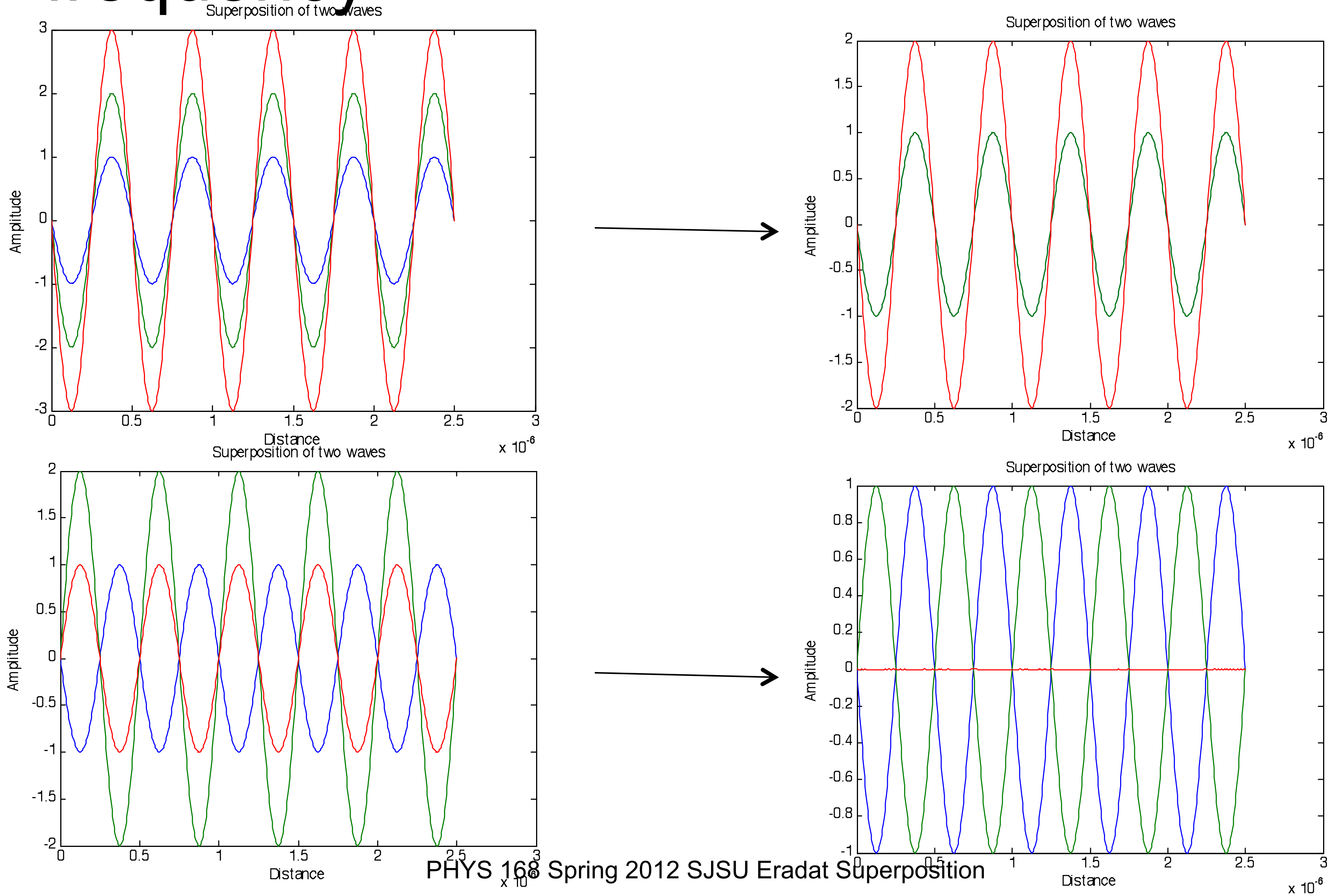
Waves in-phase interfere constructively $E = E_{\max} = (E_{01} + E_{02})^2$

Waves out of phase interfere destructively $E = E_{\min} = (E_{01} - E_{02})^2$

If $E_{01} = E_{02}$ then $E_{\max} = (2E_{01})^2$ and $E_{\min} = 0$

$\frac{\Lambda}{\lambda_0} = \frac{x_1 - x_2}{\lambda} \rightarrow$ number of waves in medium= number of waves in vacuum

Addition of two waves with same frequency



Two waves with path difference

For two waves with no initial phase difference ($\varepsilon_1 = \varepsilon_2 = 0$) but a path difference of Δx we have:

$$E_1 = E_{01}\sin\left(\omega t - k(x+\Delta x)\right) = E_{01}\sin\left[\omega t + \alpha_1\right]$$

$$E_2 = E_{02}\sin\left(\omega t - kx\right) = E_{02}\sin\left[\omega t + \alpha_2\right]$$

$$\alpha_2 - \alpha_1 = k\Delta x$$

Amplitude is a function of path difference

The resulting wave is

$$E = \boxed{2E_0\cos\left(\frac{k\Delta x}{2}\right)}\sin\left[\omega t - k\left(x + \frac{\Delta x}{2}\right)\right]$$

Constructive interference: if $\Delta x << \lambda$, or $\Delta x \approx \pm 2m\lambda$ then the resulting amplitude is ~$2E_0$

Destructive interference: $\Delta x \approx \pm(2m+1)\lambda$ then $E \approx 0$

Exercise

2.1) Plot E_1, E_2, E_1+E_2, and $(E_1+E_2)^2$ for the following two sinusoidal waves for $0<x<5\lambda$ with $\lambda=500\ nm$:

$E_1=E_{01}\sin(\omega t-(kx+\varepsilon_1))$ and $E_2=E_{02}\sin(\omega t-(kx+\varepsilon_2))$

a) same frequency, $E_{01}=E_{02}=2$, zero initial phase, both forward.

b) same frequency, $E_{01}=E_{02}=2$, $\varepsilon_1=0$, $\varepsilon_2=\pi$, both forward.

c) same frequency, $E_{01}=E_{02}=2$, $\varepsilon_1=0$, $\varepsilon_2=\pi/2$, both forward.

d) same frequency, $E_{01}=E_{02}=2$, $\varepsilon_1=0$, $\varepsilon_2=\pi$, E_1 forward, E_2 backward.

e) same frequency, $E_{02}=2E_{01}=2$, $\varepsilon_1=0$, $\varepsilon_2=0$, both forward.

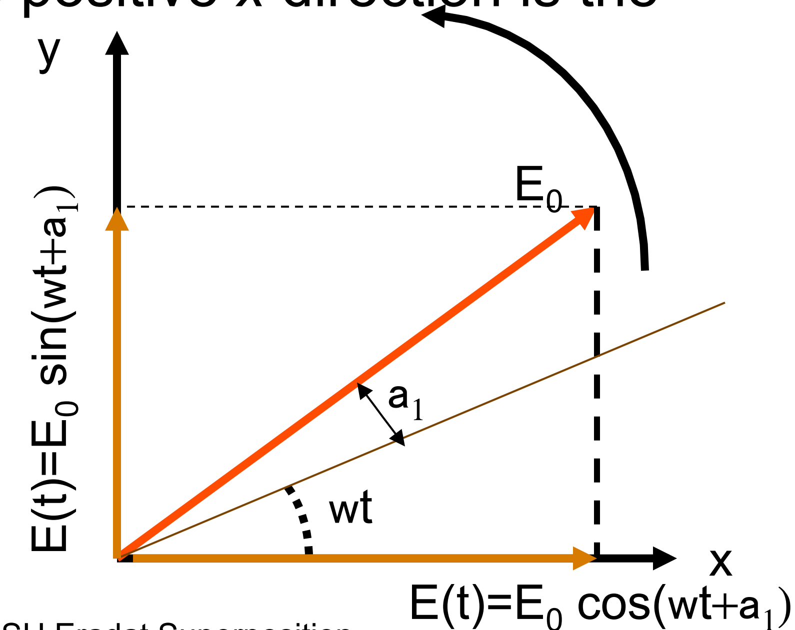
f) same frequency, $E_{02}=2E_{01}=2$, $\varepsilon_1=0$, $\varepsilon_2=\pi$, both forward.

g) Compare the results of direct superposition with the formula derived in text for case a in slide 5 (Notice the difference between $\underline{a\tan}$ and $\underline{a\tan 2}$ functions in MATLAB

Phasors and complex number representation

- Each harmonic function is shown as a rotating vector (phasor)
  - projection of the phasor on the *x* axis is the **instantaneous value of the function**,
  - length of the phasor is the maximum amplitude
  - angle of the phasor with the positive *x* direction is the **phase of the wave**.

$$E = E_0 e^{i(\omega t + \alpha_1)}$$



Example of superposition using phasors

$E_1(t) = E\cos(\omega t + \phi) \qquad E_2(t) = E\cos(\omega t)$

$\vec{\mathrm{E}}_\mathrm{p} = \vec{E}_1 + \vec{E}_2$ a vector sum of $\vec{E}_1$ and $\vec{E}_2$

Magntude of $\vec{\mathrm{E}}_\mathrm{p}$ (from triangonomety)

$\mathrm{E}_\mathrm{p}^2 = E^2 + E^2 - 2E^2\cos(\pi - \phi)$

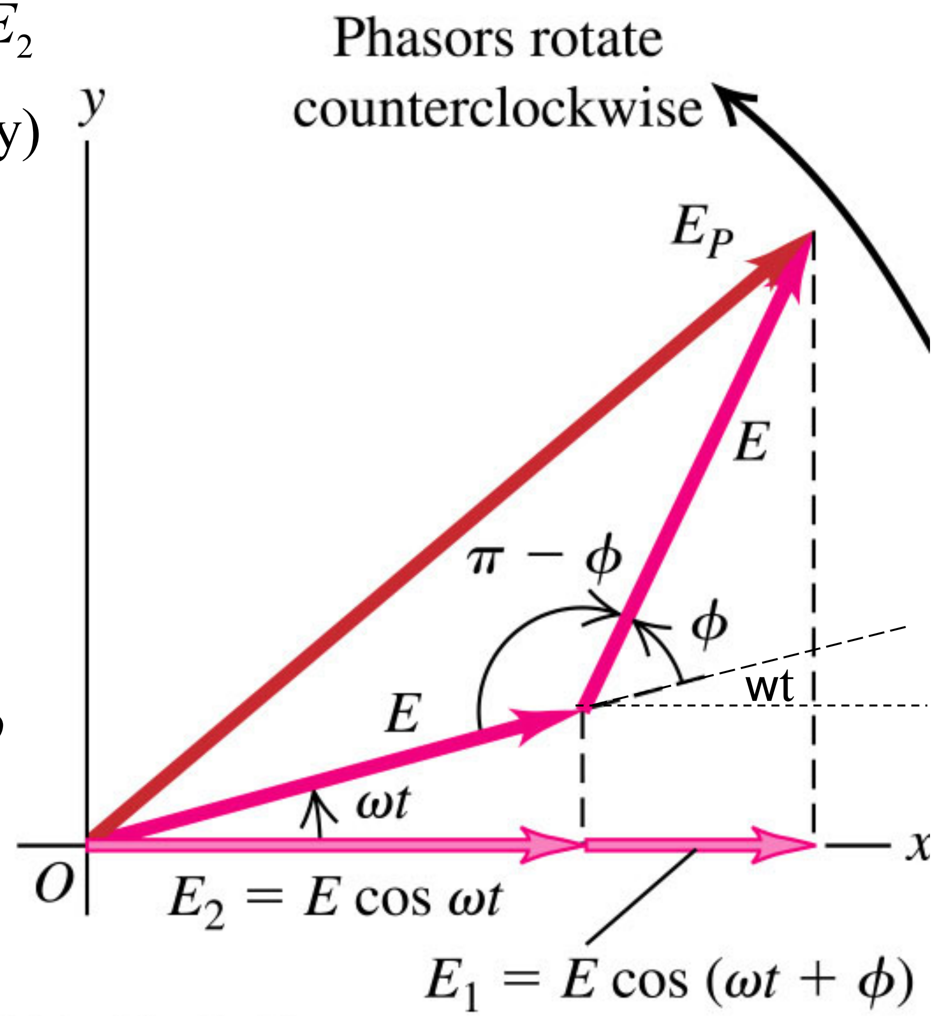
$\mathrm{E}_\mathrm{p}^2 = E^2 + E^2 + 2E^2\cos\phi$

Using $1+\cos\phi = 2\cos^2(\phi/2)$

$E_p^2 = 2E^2(1+\cos\phi) = 4E^2\cos^2(\phi/2)$

Amplitude of two indentical waves interfering with phase difference of ϕ is <u>independent of time</u>:

$$E_p = 2E\left|\cos\frac{\phi}{2}\right|$$



Superposition of many waves

Superposition of any number of coherent harmonic waves with a given frequency, ω and traveling in the same direction leads to a harmonic wave of that same frequency.

$$E = \sum_{i=1}^{N} E_{0i} \cos(\alpha_i \pm \omega t) = E_0 \cos(\alpha \pm \omega t)$$

$$E_0^2 = \sum_{i=1}^{n} E_{0i}^2 + 2\sum_{j>i}^{N} \sum_{i=1}^{N} E_{0i} E_{0j} \cos(\alpha_i - \alpha_j) \quad \text{and} \quad \tan\alpha = \frac{\sum_{i=1}^{N} E_{0i} \sin\alpha_i}{\sum_{i=1}^{N} E_{0i} \cos\alpha_i}$$

$\alpha_i = -(kx + \varepsilon_i)$ and $\alpha_j = -(kx + \varepsilon_j)$

For coherent sources $\alpha_{\mathrm{i}} = \alpha_j$ and $E_0^2 = \sum_{i=1}^{N} E_{0i}^2 + 2\sum_{j>i}^{N} \sum_{i=1}^{N} E_{0i} E_{0j} = \left(\sum_{i=1}^{N} E_{0i} \right)^2$

For incoherent sources (random phases) the second term is zero.

Flux density for N equal-amplitude emmiters: $\left(E_0^2\right)_{incoh} = NE_{01}^2$; $\left(E_0^2\right)_{coh} = \left(NE_{0i}\right)^2$

Exercise

2.2) Write a MATLAB routine to calculate the amplitude and phase of N harmonic waves (cosine) with same frequencies but varying initial phase and amplitudes. Assume the wavelength is $500\ nm$ and $V = c$

a) The program should read the phase and amplitude of the waves from a file that has two columns and N rows. Test the program for the following waves $E_1 = 1,\ \varepsilon_1 = 0,\ E_2 = 1,\ \varepsilon_2 = \pi/4$. Once made sure it is working, create a file with the folloing waves and plot their superposition from 0 to 5λ.

$$E_1 = 1,\ \varepsilon_1 = 0,\ E_2 = 1,\ \varepsilon_2 = 10,\ E_3 = 2,\ \varepsilon_3 = 20^0,\ E_4 = 3,$$
$$\varepsilon_4 = 30^0,\ E_5 = 2,\ \varepsilon_5 = 40^0,\ E_6 = 1,\ \varepsilon_6 = 50^0,\ E_7 = 1,\ \varepsilon_7 = 60^0$$

b) Next run the program for $N = 101$ and $\varepsilon_i = \varepsilon_1 + \frac{i}{100}\pi$, where $\varepsilon_1 = \frac{\pi}{2}$ and $E_i = 2$. This time create the phases and amplitudes inside the routine and don't read from a file.

Addition of waves: different frequencies I

Mathematics behind light modulation and light as a carrier of information. Two propagating waves are superimposed

$$E_1 = E_{01}\cos(k_1 x - \omega_1 t)$$

$$E_2 = E_{01}\cos(k_2 x - \omega_2 t)$$

$k_1 > k_2$ and $\omega_1 > \omega_2$ with equal amplitudes and zero inital phases

$$E = E_1 + E_2 = E_{01}[\cos(k_1 x - \omega_1 t) + \cos(k_2 x - \omega_2 t)]$$

using $\cos\alpha + \cos\beta = 2\cos\frac{1}{2}(\alpha+\beta)\cos\frac{1}{2}(\alpha-\beta)$

$$E = 2E_{01}\cos\frac{1}{2}\left[(k_1 + k_2)x - (\omega_1 + \omega_2)t\right] \times \cos\frac{1}{2}\left[(k_1 - k_2)x - (\omega_1 - \omega_2)t\right]$$

Need to simplify this

Addition of waves: different frequencies II

$E = 2E_{01}\cos\left[k_m x - \omega_m t\right] \times \cos\left[\bar{k}x - \bar{\omega}t\right]$ with the following definitions

Average angular frequency $\equiv \bar{\omega} = \left(\omega_1 + \omega_2\right)/2$

Average propagation number $\equiv \bar{k} = \left(k_1 + k_2\right)/2$

Modulation angular frequency $\equiv \omega_m = \left(\omega_1 - \omega_2\right)/2$

Modulation propagation number $\equiv k_m = \left(k_1 - k_2\right)/2$

Time-varying modulation amplitude $\equiv E_0(x,t) = 2E_{01}\cos\left[k_m x - \omega_m t\right]$

Superimposed wavefunction: $\underline{E = E_0(x,t)\cos\left[\bar{k}x - \bar{\omega}t\right]}$

For large ω if $\omega_1 \approx \omega_2$ then $\bar{\omega} >> \omega_m$ we will have a slowly varying amplitude with a rapidly oscillating wave

Irradiance of two superimposed waves with different frequencies

$$E_0^2(x,t) = 4E_{01}^2 \cos^2\left[k_m x - \omega_m t\right] = 2E_{01}^2\left[1 + cos\left(2k_m x - 2\omega_m t\right)\right]$$

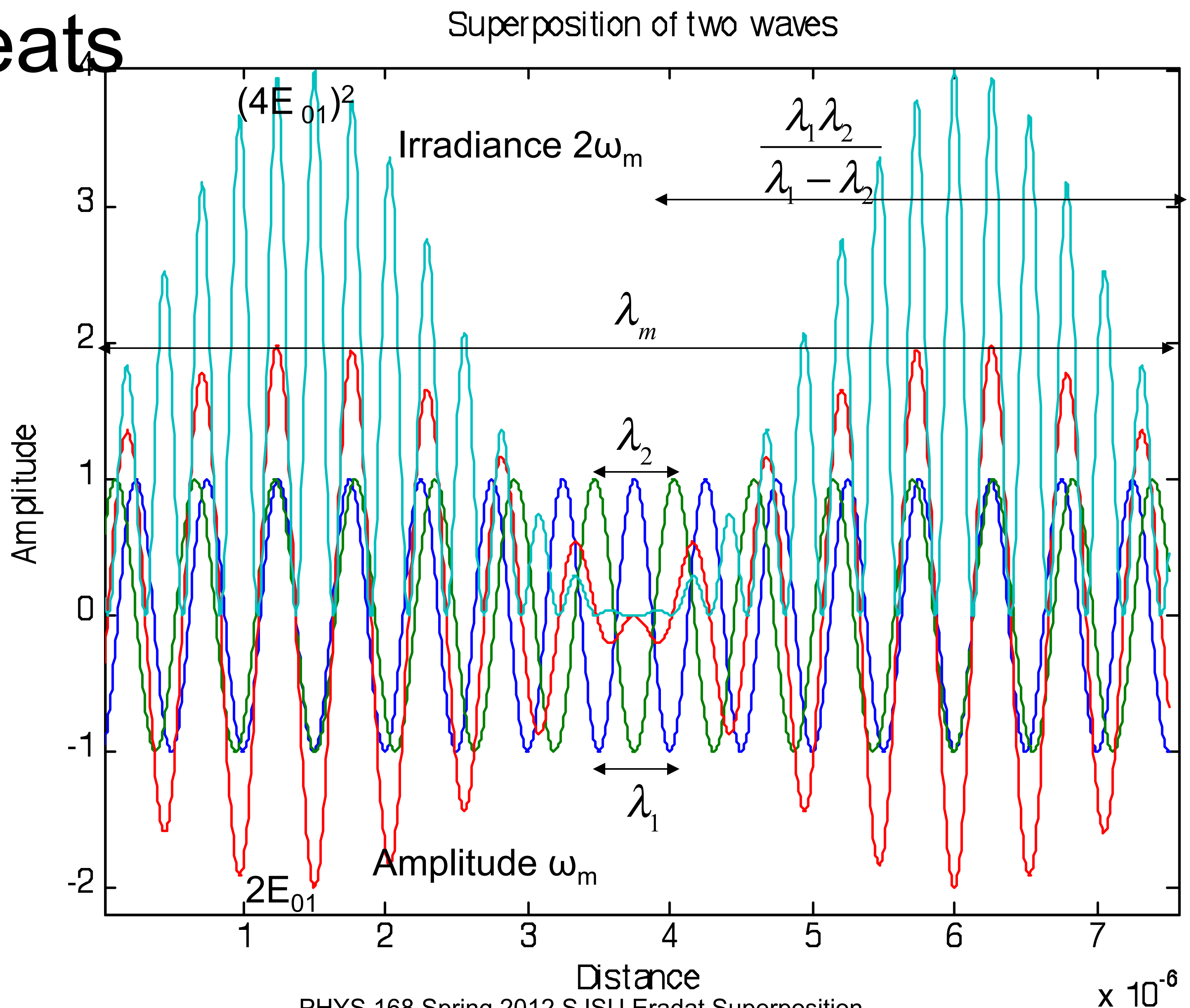
Beat frequency $\equiv 2\omega_m = \omega_1 - \omega_2$ or oscillation frquency of the $E_0^2(x,t)$

Amplitude, E_0, oscilates at ω_m, the modulation freuency

Irradiance, E_0^2, varies at $2\omega_m$, twice the modulation frequency

Two waves with different amplitudes produce beats with less contrast.

Beats


Superposition of two waves
$(4E_{01})^2$
Irradiance $2\omega_m$
$\frac{\lambda_1\lambda_2}{\lambda_1-\lambda_2}$
λ_m
λ_2
λ_1
Amplitude ω_m
$2E_{01}$
Amplitude
Distance
x 10^{-6}
4
3
2
1
0
-1
-2
1
2
3
4
5
6
7

Phase velocity of a wave

Constant-phase point: a point on a progressive wave with a constant magnitude of the disturbance.

$$\psi(x,t) = A\sin(kx \mp \omega t + \varepsilon)$$

Phase velocity of a wave is speed of the motion of a constant-phase point on a disturbance $V_{phase} = \left.\frac{\partial x}{\partial t}\right|_{\phi}$

Phase velocity is speed of the motion of the disturbance.

Ex: calculate value of the phase velocity for the above wave.

Phase of a harmonic wave at time t: $\phi = kx \mp \omega t + \varepsilon$

For constant phase: $\frac{\partial \phi}{\partial t} = 0 \rightarrow k\left.\frac{\partial x}{\partial t}\right|_{\phi} \mp \omega = 0 \rightarrow \boxed{V_{phase} == \pm\frac{\omega}{k}}$

Group velocity

In nondispersive media velocity of a wave is independent of its frequency.

For a single frequency wave there is one velocity and that is $\boxed{V_{phase} = \frac{\omega}{k}}$

When a vave is composed of different frequency elements, the resulting disturbance will travel with differnt velocity than phase velocity of its components.

$$E = 2E_{01}\cos\left[k_m x - \omega_m t\right] \times \cos\left[\bar{k}x - \bar{\omega}t\right]$$

$V_{phase} = \frac{\bar{\omega}}{\bar{k}}$ velocity of a constant phase point on the high frequency wave

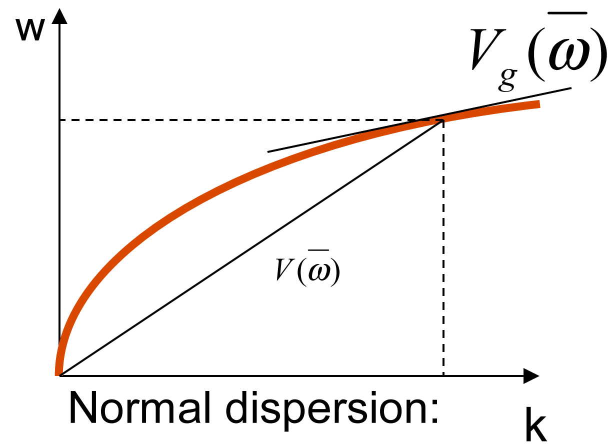
$V_{group} = \frac{\omega_m}{k_m} = \left(\frac{d\omega}{dk}\right)_{\bar{\omega}}$ velocity of a constant amplitude point on the modulation envelope

V_g may be smaller, equal, or larger than V_p

To calculate the V_p and V_g we need the dispersion relation $\omega = \omega(k)$

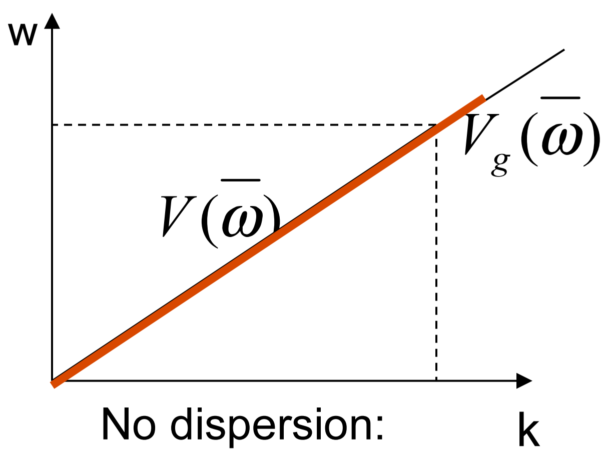
Dispersion relation ω v.s. k or $\omega = f(k)$

- **Phase velocity** for a given frequency is <u>slope of a line on the dispersion curve</u> that connects that point to the origin or w/k.
- **Group velocity** for that frequency is the <u>slope of the dispersion curve at that point</u> or dw/dk.
- We also may have a **gap** in the dispersion relation for a **frequency band**. In that case the velocities are **not defined** because waves can not propagate



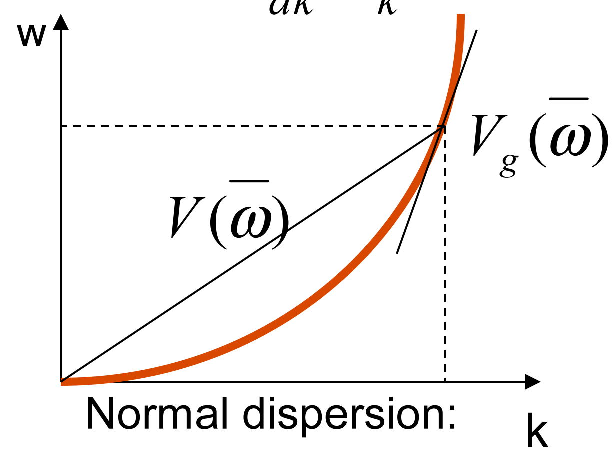
Normal dispersion:

$V_g < V_p$ or $\frac{d\omega}{dk} < \frac{\omega}{k}$



No dispersion:

$V_g = V_p$ or $\frac{d\omega}{dk} = \frac{\omega}{k}$



Normal dispersion:

$V_g > V_p$ or $\frac{d\omega}{dk} > \frac{\omega}{k}$

Finite waves

- Finite wave: any wave starts and ends in a certain time interval
- Any finite wave can be viewed as a really long pulse
- Any pulse is a result of superposition of numerous different frequency harmonic waves called **Fourier components.**
- **Wave packet** is a localized pulse that is composed of many waves that cancel each other everywhere else but at a certain interval in space.
- We need to study Fourier Analysis to understand actual waves, pulses, and wave packets.
- **Width of a wave packet** is proportional to the **range of k_m** of the wave packet.
- Since each component of the wave packet has different phase velocity in the medium, through the relationship $V_p=w/k$, k of the components change in the dispersive media.
- As a result **k_m of the modulation disturbance changes**
- **Consequently group velocity changes.**
- This results in change of the width of the wave packet.
- So wave packets inside a medium may spread or become narrower.