Fourier Integrals

Nonperiodic waves
Fourier Integrals

Nonperiodic waves

- In optics and quantum mechanics all real waves are pulses.
- In order to generate a pulse out of harmonic functions that have a certain width and shape, we need to know
  - what frequency elements to add
  - How much of each frequency element to add

Addition of waves: different frequencies

To generate beats we added two frequencies ω_1 and ω_2

$$E = 2E_{01}\cos\left[k_m x - \omega_m t\right]\times\cos\left[\bar{k}x - \bar{\omega}t\right]$$

the carrier frequency is average of the added frequencies $\bar{\omega} = \frac{1}{2}\left(\omega_1 + \omega_2\right)$

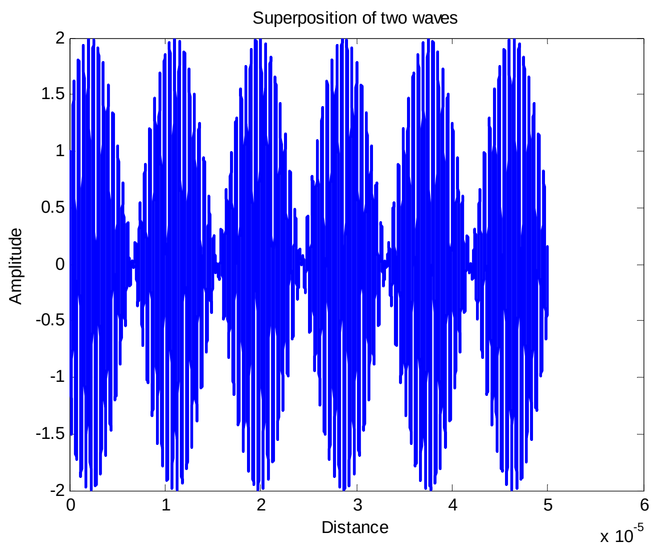
1) If we add more frequency elements symmetrically around $\bar{\omega}$, then $\bar{\omega}$ will not change.

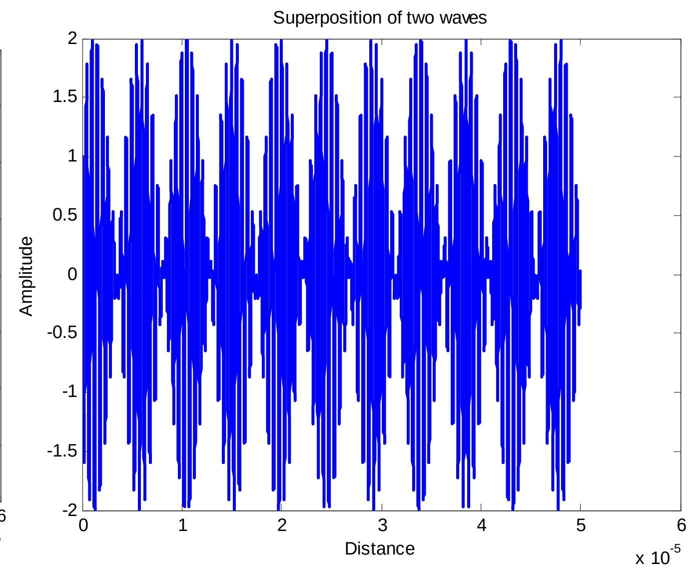
2) If we reduce the spacing between the frequency elements, then frequency of modulation $\omega_m = \frac{1}{2}\left(\omega_1 - \omega_2\right)$ will decrease. This is the frequency of the envelope meaning the beats will have more separation.

3) When number of frequency elements go to infinity, we will have a single pulse. The λ_m goes to infinity.

Beats with different frequency intervals

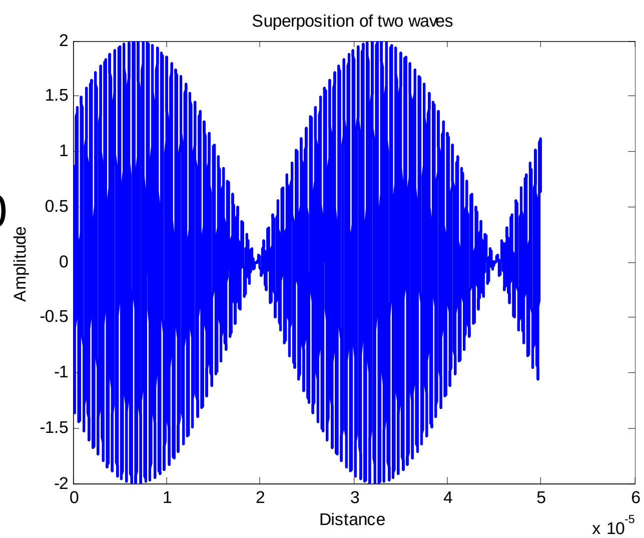
$\Delta\lambda$=30

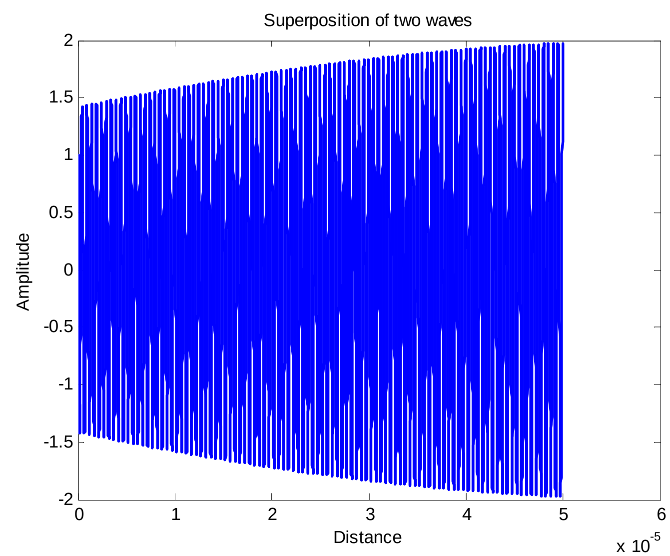




$\Delta\lambda$=60

$\Delta\lambda$=10





$\Delta\lambda$=1

Square Wave

For the following square wave

$$f(x)=\begin{cases}+1 & -\lambda/a<x\le\lambda/a \\ 0 & \lambda/a<x\le 3\lambda/a\end{cases} \quad \text{even}$$

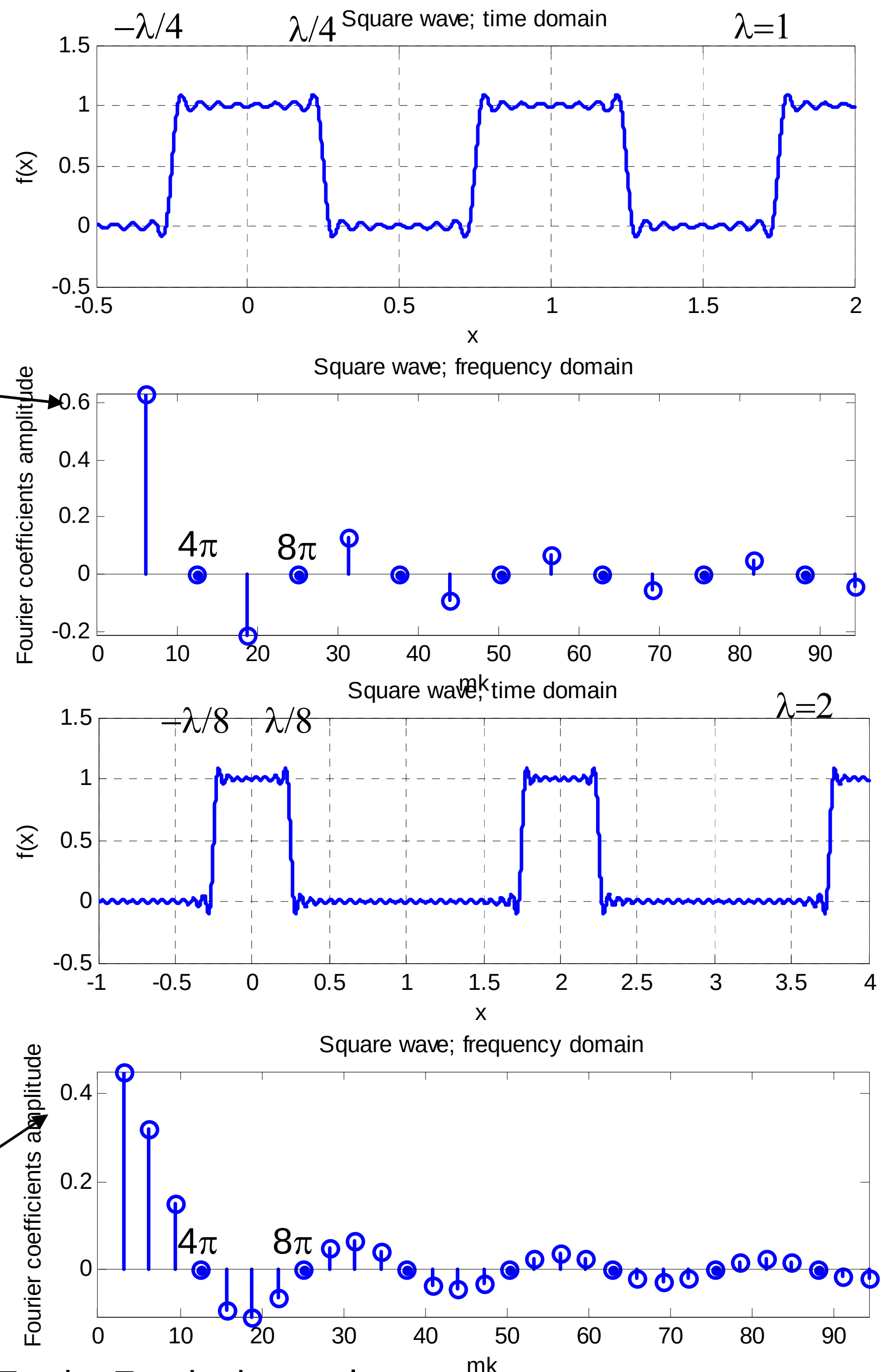
The Fourier components are

$$f(x)=\frac{2}{a}+\sum_{m=1}^{\infty}\frac{4}{a}\operatorname{sin}c\left(\frac{m2\pi}{a}\right)\cos(mkx)$$

Frequency spectrum is amplitude of each frequency component in $f(x)$ vs. k. It expresses weighing factors of each harmonic component at any spatial frequency present in the synthesis.

Let's keep width of the peaks constant while increasing the wavelength.

Zeros of the sinc function happen at constant mk. As k gets smaller because of large wavelength, m, the number of harmonics gets larger.

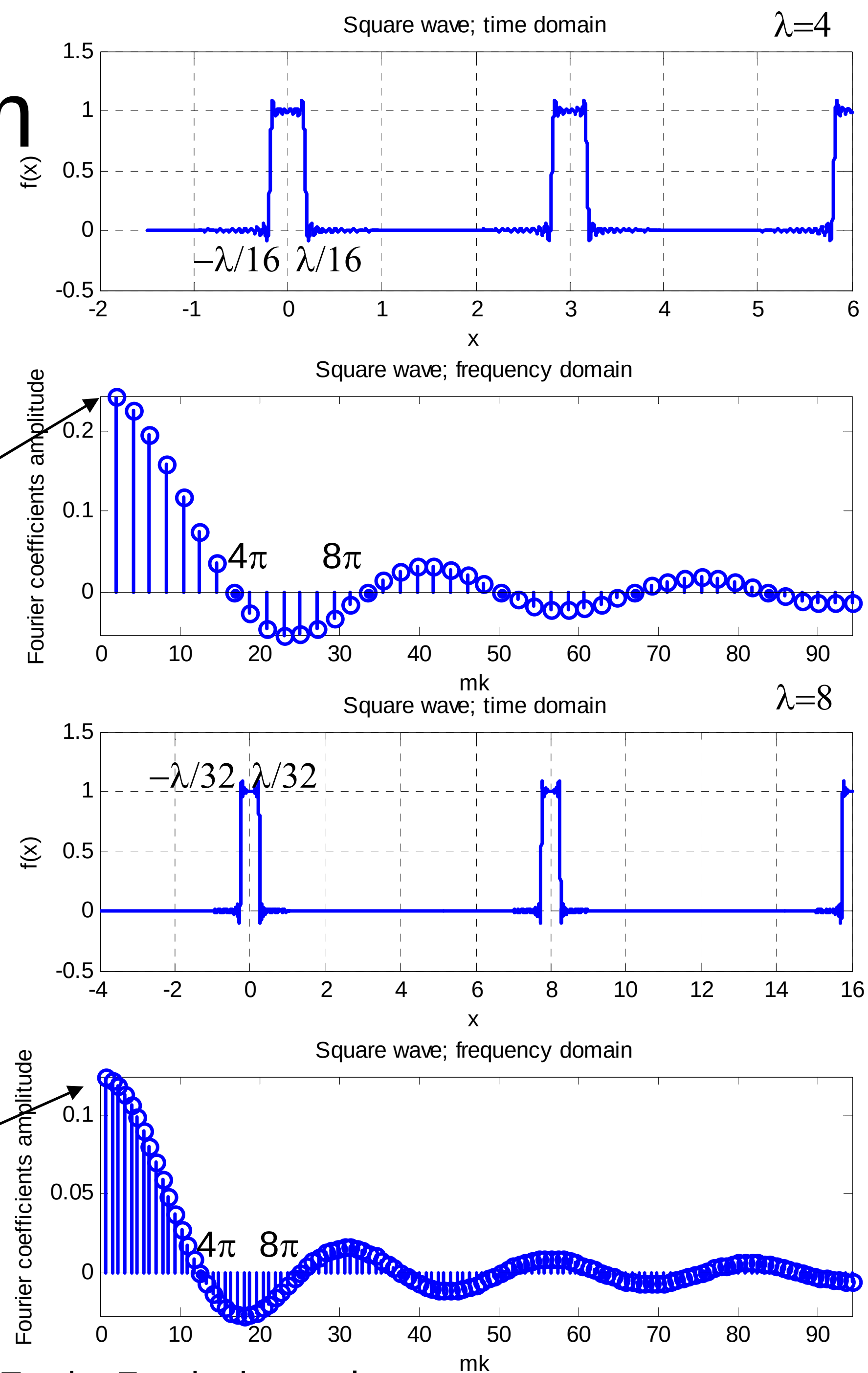


Frequency spectrum of a pulse

As we increase the separation between the pulses, the Fourier components get closer in the frequency space.

As the <u>wavelength goes to infinity</u>, the wave gets closer to expression of a pulse, the Fourier components get closer and leading to a continuum.

At the limit we have a <u>pulse in time domain</u> and a <u>continuous Fourier series in frequency domain</u> known as <u>Fourier integral</u>



Fourier Transforms

As $\lambda \to \infty$ or k $\to$ 0 the Fourier series transforms to Fourier integrals

Finite $\lambda \to$ $$f(x) = \sum_{m=0}^{\infty} A_m \cos(mkx) + \sum_{m=0}^{\infty} B_m \sin(mkx)$$

Infinite $\lambda \to$ $$f(x) = \lim_{k\to 0}\left(\sum_{m=0}^{\infty} A_m \cos(mkx)\Delta k + \sum_{m=0}^{\infty} B_m \sin(mkx)\Delta k \right)$$

Infinite $\lambda \to$ $$\boxed{f(x) = \frac{1}{\pi}\left[\int_0^{\infty} A(k)\cos(kx)dk + \int_0^{\infty} B(k)\sin(kx)dk \right]}$$

Using the orthognality of sine and cosine functions we can find:

$$\boxed{A(k) = \int_{-\infty}^{\infty} f(x)\cos(kx)dx}; \qquad \boxed{B(k) = \int_{-\infty}^{\infty} f(x)\sin(kx)dx}$$

$A(k)$ and $B(k)$ are called Fourier cosine and sine transforms of the function $f(x)$.

Complex representation of the Fourier transforms

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(k) e^{-ikx} dk$$

$$F(k) = \int_{-\infty}^{\infty} f(x) e^{ikx} dx$$

The function $F(k)$ is spoken of as the Fourier transform of $f(x)$

$$F(k) = A(k) + iB(k) = \mathcal{F}\left\{ f(x) \right\}$$

The sine and cosine transforms are:

$$\mathcal{F}\left\{ f(x) \right\} = \mathcal{F}_C\left\{ f(x) \right\} + i\mathcal{F}_S\left\{ f(x) \right\}$$

And inverse Fourier transform of $F(k)$ is:

$$f(x) = \mathcal{F}^{-1}\left\{ F(k) \right\}$$

Two dimensional Fourier Transform

$$f(x,y)=\frac{1}{(2\pi)^2}\iint_{-\infty}^{\infty}F\left(k_x,k_y\right)e^{-i(k_x x+k_y y)}dk_x dk_y$$

$$F(k_x,k_y)=\iint_{-\infty}^{\infty}f\left(x,y\right)e^{i(k_x x+k_y y)}dxdy$$

Gibbs Phenomena

- Overshoot of a synthesized f(x) by 9% of the amplitude at discontinuities, known as Gibbs Phenomena, is due to the limited number of Fourier components used to create f(x).
- When N actually goes to infinity, f(x) would be 100% accurate.
- For limited N there is oscillations in f(x) with frequency of Nf_0.
- When the N is large enough the width of the oscillations $1/2Nf_0$ goes to zero and the overshoot contains zero power.
- This allows usage of Fourier series even if there is a discrepancy with the actual function.

Exercise

4.1)

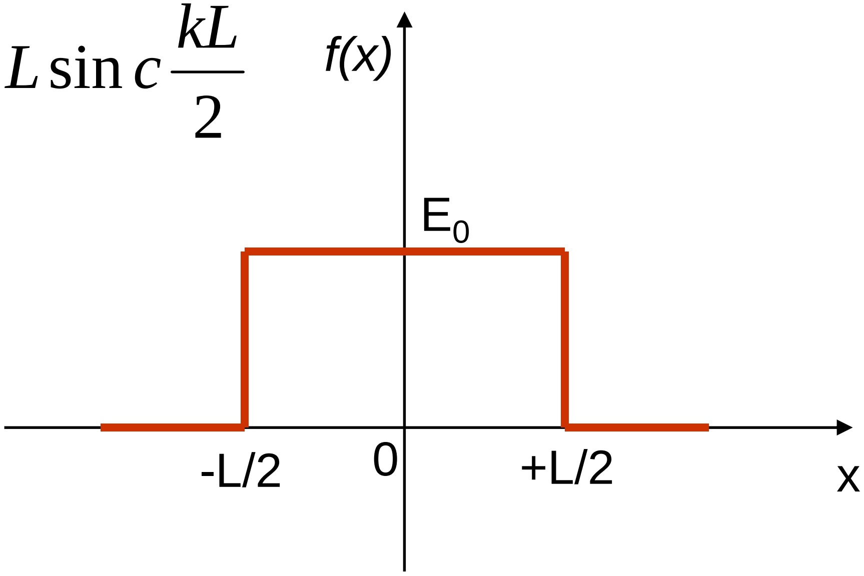
a) Calculate Fourier transform of the square pulse given in this figure.

b) Plot the $\mathcal{F}\left\{f\left(x\right)\right\}$

c) Use FFT function in MATLAB to plot the $\mathcal{F}\left\{f\left(x\right)\right\}$

What happens to the zeros of $\mathcal{F}$ as L increases?

Ans: $\mathcal{F}\left\{f\left(x\right)\right\} = \mathcal{F}_C\left\{f\left(x\right)\right\} = E_0 L \sin c\,\frac{kL}{2}$



Exercise

4.2)

a) Calculate complex Fourier transform of the function given in this figure.

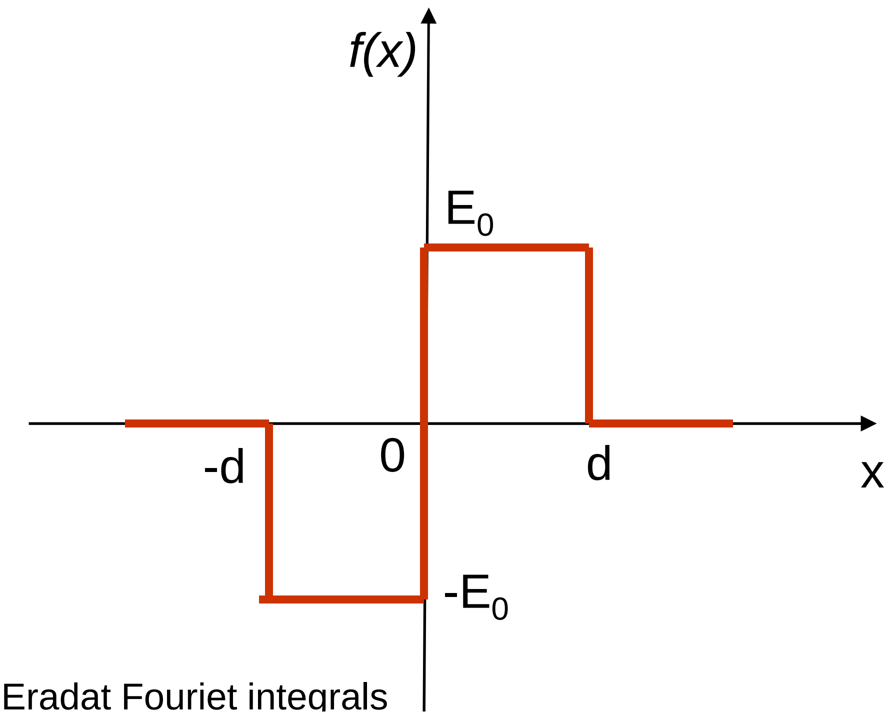
b) Plot the $\mathcal{F}\left\{f(x)\right\}$

c) Use FFT function in MATLAB to plot the $\mathcal{F}\left\{f(x)\right\}$

Ans: $\mathcal{F}\left\{f(x)\right\} = i\mathcal{F}_S\left\{f(x)\right\} = 2iE_0 d\,\dfrac{\sin^2(kd/2)}{kd/2}$

Use $\mathcal{F}\left\{f(x)\right\} = \mathcal{F}_C\left\{f(x)\right\} + i\mathcal{F}_S\left\{f(x)\right\}$ or the exponential representation

Note $\mathcal{F}_C$ and $\mathcal{F}_S$ are real.



Exercise

4.3) a) Determine Fourier transform of the wave train given by $E_1(x) = P(x)\cos k_p x$ and $E_2(x) = P(x)\cos^2 k_p x$ where $P(x)$ is the unit square pulse. k_p is the spatial frequency of the oscillatory region of the pulse.

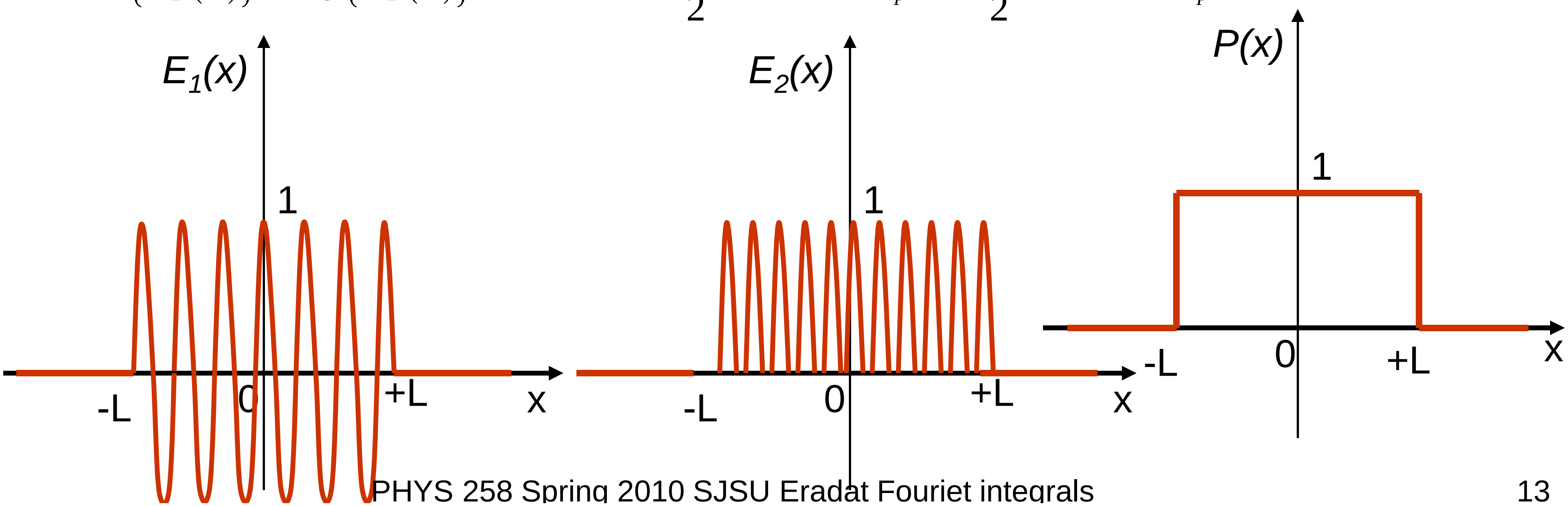
b) Plot the Fourier transform for both E_1 And E_2.

c) Sketch the transforms in the limit as width of the $P(x)$ extend to infinity.

d) Use FFT function in MATLAB to plot the $\mathcal{F}\{E(x)\}$ for both functions.

Ans: $\mathcal{F}\{E_1(x)\} = \mathcal{F}_C\{E_1(x)\} = L[\sin c(k_p + k)L + \sin c(k_p - k)L];$

$$\mathcal{F}\{E_2(x)\} = \mathcal{F}_C\{E_2(x)\} = L\sin ckL + \frac{L}{2}\sin c(k + 2k_p)L + \frac{L}{2}\sin c(k - 2k_p)L;$$



Time domain Fourier transform

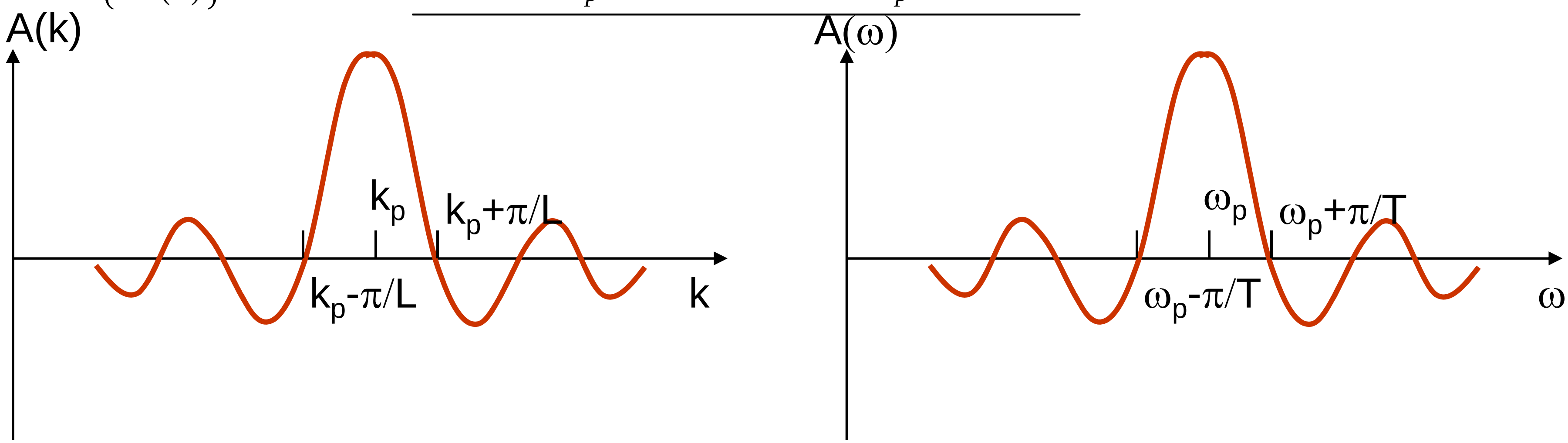
The same wavepacket of exercise 4.3 in time domain:

$$E(t) = \begin{cases} E_0 \cos \omega_p t & -T \le t \le T \\ 0 & |t| > T \end{cases}$$

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{-i\omega t} d\omega$$

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{i\omega t} dt$$

$$\mathcal{F}\{E(t)\} = A(\omega) = \underline{T[\sin c(\omega_p + \omega)T + \sin c(\omega_p - \omega)T];}$$



Frequency bandwidth

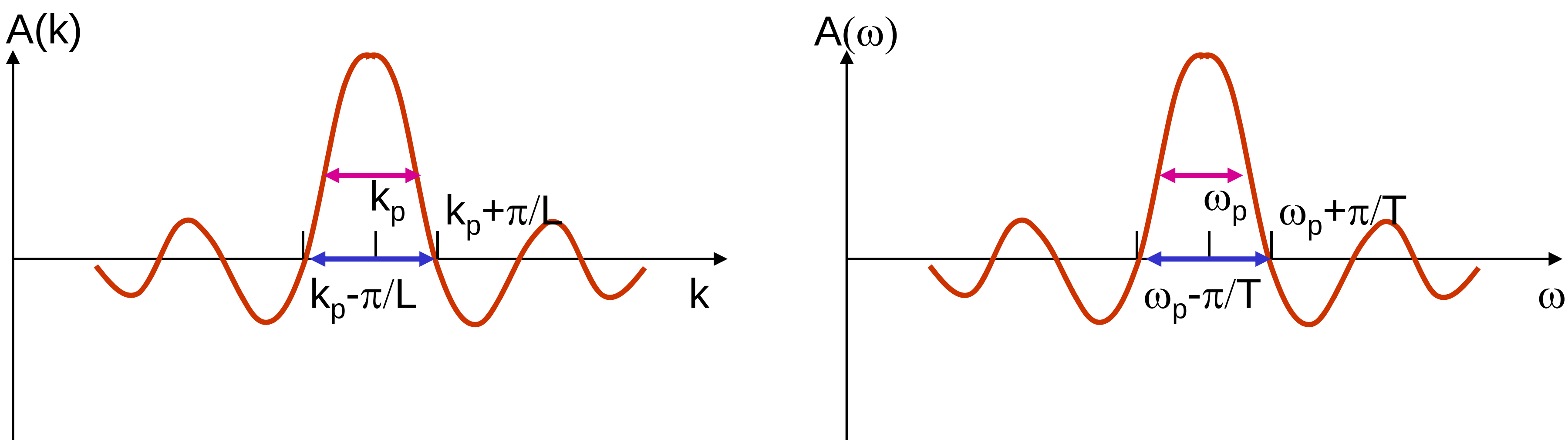
In time domain temporal width of the pulse is $\Delta t = 2T$ while $0 < \omega < \infty$

and width of the transform is said to be $\Delta\omega \approx \dfrac{2\pi}{T}$. Thus $\Delta t \Delta\omega = 4\pi$

Product of the width of the pakage in t-space and ω-space is constant.

In frequency domain spatial width of the pulse is $\Delta x = 2L$ while $0 < k < \infty$

and width of the transform is said to be $\Delta k \approx \dfrac{2\pi}{L}$. Thus $\Delta x \Delta k = 4\pi$



Uncertainty relations

Product of the width of the pakage in k-space and x-space is constant.

$\Delta\omega$, $\Delta\nu = \Delta\omega / 2\pi$ and $\Delta k = 2\pi\Delta\lambda / \lambda^2$ are known as frequency bandwidths.

Choice of $\Delta\omega$ and Δk are somewhat arbitrary.

Important fact is that $\Delta t \Delta\nu \approx 1$ and $\Delta x \Delta k =$ constant

These relations are known as uncertainty relations and have profound practical importance. They impose limits on accuracy of our measurements or precision achievements.

Choosing a frequency bandwidth of $\Delta\nu$ restricts us in receiving signals that are shorter than or $\Delta t = \dfrac{1}{\Delta\nu}$

Coherence length

If a wavetrain has frequency bandwidth of $\Delta\nu,$ it has been produced within $\Delta t \approx 1/\Delta\nu$ time interval from a group of oscillators. These oscillators may have produced waves that have constant phase relationship with each other only during Δt.

The next wavetrain has a completely different phase relationship.

Δt_c is known as coherence time of the source.

$\Delta l_c = c\Delta t_c$ is known as the coherence length of the light produced by the source.

$\Delta\nu$ is due to natural linewidth of the plus which is due to different broadening mechanisms such as thermal (Doppler effect), collision etc.

Meaning of negative spatial frequency

- Complex representation of Fourier transforms gives rise to a symmetrical distributions of positive and negative spatial frequency terms.
- Certain optical phenomena such as diffraction occur symmetrically in space.
- A relationship between these phenomena and spatial frequency spectrum can be constructed if we use both negative and positive spatial frequency terms.
- Negative frequency becomes a useful mathematical device to describe physical systems that are symmetrical around a central point

About the constants of the Fourier transforms

$$f(x) = \alpha \int_{-\infty}^{\infty} F(k) e^{-ikx} dk$$

$$F(k) = \beta \int_{-\infty}^{\infty} f(x) e^{ikx} dx$$

α and β can be anything as long as

$\alpha\beta = \frac{1}{2\pi}$ for the 1D Fourier transforms

$\alpha\beta = \frac{1}{4\pi^2}$ for the 2D Fourier transforms

When $\alpha = \beta$ we have symmetric Fourier transformation.

Parseval's identities for Fourier integrals

Sum of the squares of the Fourier coefficients of a function is equal to the square integral of the function.

For Fourier series: $\frac{1}{L}\int_{-L}^{L}\left\{ f(x)\right\}^2 dx = \frac{A_0^2}{2} + \sum_{m=1}^{\infty}\left(A_m^2 + B_m^2 \right)$

If $F_S(k)$ and $G_S(k)$ are Fourier sine transforms of $f(x)$ and $g(x)$, then

$$\int_0^\infty F_s(k)G_s(k)dk = \int_0^\infty f(x)g(x)dx$$

If $F_C(k)$ and $G_C(k)$ are Fourier cos transforms of $f(x)$ and $g(x)$, then

$$\int_0^\infty F_C(k)G_C(k)dk = \int_0^\infty f(x)g(x)dx$$

If $f(x) = g(x)$ then

$\int_0^\infty \left\{F_s(k)\right\}^2 dk = \int_0^\infty \left\{ f(x)\right\}^2 dx$ and $\int_0^\infty \left\{F_C(k)\right\}^2 dk = \int_0^\infty \left\{ f(x)\right\}^2 dx$

For general Fourier transforms:

$$\int_{-\infty}^\infty F(k)G(k)^* dk = \int_{-\infty}^\infty f(x)g(x)^* dx$$

$G(k)^*$ is the complex conjugate of the $G(k)$

The convolution theorem

If $F(k)$ and $G(k)$ are the Fourier transforms of $f(x)$ and $g(x)$ then

$$\int_{-\infty}^{\infty} F(k)G(k)e^{-ikx}dk = \int_{-\infty}^{\infty} f(u)g(x-u)du$$

If we show the *convolution* of the functions f and g with $f*g$,

$$f*g = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(u)g(x-u)du$$

$$\mathcal{F}\left\{f*g\right\} = \mathcal{F}\left\{f\right\}\mathcal{F}\left\{g\right\}$$

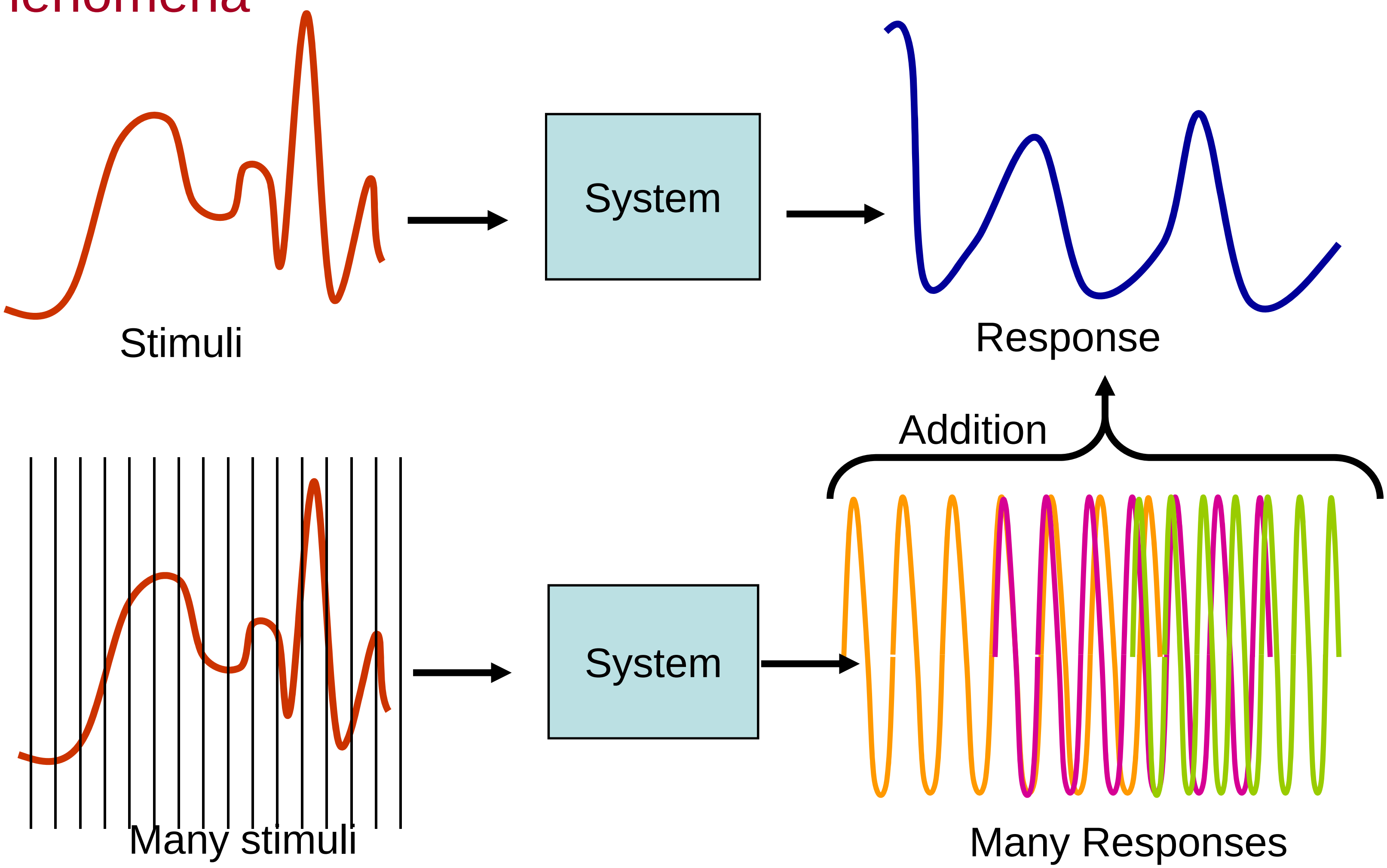
<u>Fourier transform of the convolution</u> of two functions is equal to the <u>product of their Fourier transforms</u>.

Discrete Fourier transform

- Analytical Fourier transformation is possible for some functions.
- If there is no functional representation of data such as image of a person or collection of data points, then how we perform Fourier analysis?
- There is a numerical techniques to determine frequency content of such data known as **discrete Fourier transform**.

Analysis of two-dimensional Signals and systems

- Linearity is a common property of many physical phenomena



Linear systems theory

- Optical imaging operation is a linear mapping of object light distributions to image light distributions.
- This mapping is done by the wave equation.
- Linear systems theory gives us the ability to express the response to a complicated stimulus in terms of the responses to certain elementary stimuli.
- Some mathematical tools are used in describing linear phenomena

Two dimensional Fourier transformation

Goodman's notation: f_x is equivalent to k_x spatial frequency in x diretion

Fourier transform or Fourier spectrum of a complex-valued function $g(x,y)$ with two independent variables x,y (space) is:

$$G(f_x,f_y)=\mathcal{F}\{g(x,y)\}=\int\int_{-\infty}^{\infty} g(x,y)e^{-j2\pi(f_x x+f_y y)}dxdy$$

The transform itself is a complex-valued function of two independent variables f_X, f_Y (frequency)

The inverse Fourier transform of $G(f_X, f_Y)$

$$\mathcal{F}^{-1}\{G\}=\int\int_{-\infty}^{\infty} G(f_X,f_Y)e^{j2\pi(f_x x+f_y y)}df_x df_y$$

or Fourier integral representaion of function $g(x,y)$

Fourier transformation as a decomposition of g(x,y) in 2D

In dealing with linear systems we need to decompose a complicated input to a number of simple elements. Fourier transformation does this job.

$$g(x) = \mathcal{F}^{-1}\{G\} = \int_{-\infty}^{\infty} \underbrace{G\left(f_X\right)}_{\substack{\text{Weighting}\\ \text{factor or}\\ \text{Amplitudes}}} \underbrace{e^{j2\pi f_X x}}_{\substack{\text{Elementary}\\ \text{function}}} \, df$$

This is expressing the space function $g(x)$ in terms of its frequency spectrum $G\left(f_X\right)$.

This is linear combination of the elementry functions of the form $e^{j2\pi f_X x}$

The $G(f)$s are the weighting factors or amplitudes of each function.

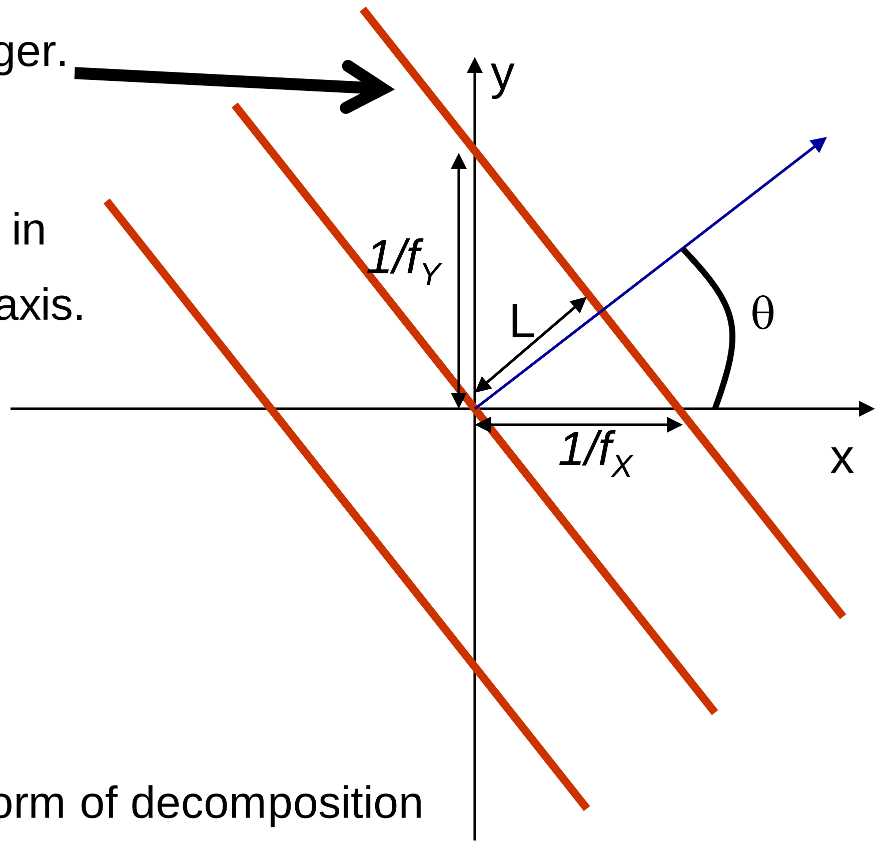
Decomposition of g(x,y) in 2D

Goal: Finding orientation and frequency of the constant phase lines for exponential elementary functions.

For $2D$ Fourier transforms the elementary functions have the form: $e^{j2\pi(f_X x+f_Y y)}$

For each frequency pair (f_X, f_Y) the corresponding elementary function has a zero or $2\pi m$ phase along the lines described by:

$$y = \underbrace{-\frac{f_X}{f_Y}}_{\text{Slope of the line perpendicular to the constant phase plane}} x + \frac{n}{f_Y}$$ where n is an integer.



So the elementary functions are being directed in (x, y) plane at an angle θ with respect to the x axis.

$$\theta = \tan^{-1}\left(\frac{f_X}{f_Y}\right)$$

and spatial period of $L = \frac{1}{\sqrt{f_X^2 + f_Y^2}}$.

(Use $\cos\theta = \frac{L}{1/f_X}$ & $\sin\theta = \frac{L}{1/f_Y}$ to find L)

So we see the inverse Fourier transform as a form of decomposition with a periodic nature of the exponential elementary functions.

Fourier transformation and existence conditions

Existence conditions (sufficient):

1) g must be absolutely integrable over the finite (x,y) plane

2) g must have only finite number of discontinuitiesand a finite number of maxima and minima.

3) g must have no infinite discontinuities.

If a function does not satisfy all of the above conditions and yet can be written as sum of the functions that satisfy the conditions, we can Fourier transform it by taking the Fourier transform of the pieces. Limit of this new sequence is called generalized Fourier transform of the function.

If the transform exists, then we use it and don't worry about the existanace conditios.

Fourier Transforms (summary)

$1D$ Hecht) $\quad g(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(k)e^{-ikx}dk \qquad F(k) = \int_{-\infty}^{\infty} f(x)e^{ikx}dx$

$2D$ Hecht) $\quad f(x,y) = \frac{1}{(2\pi)^2}\iint_{-\infty}^{\infty} F(k_x,k_y)e^{-i(k_x x + k_y y)}dk_x dk_y$

$2D$ inverse Hecht) $\quad F(k_x,k_y) = \iint_{-\infty}^{\infty} f(x,y)e^{i(k_x x + k_y y)}dxdy$

$2D$ Goodman) $\quad g(x,y) = \mathcal{F}^{-1}\{G\} = \iint_{-\infty}^{\infty} G(f_X, f_Y)e^{j2\pi(f_X x + f_Y y)}df_X df_Y$

$2D$ inverse Goodman) $G(f_X, f_Y) = \mathcal{F}\{g(x,y)\} = \iint_{-\infty}^{\infty} g(x,y)e^{-j2\pi(f_X x + f_Y y)}dxdy$

Note: the reason Goodman's notation does not have the coefficients $1/(2\pi)^2$ is that he uses spatial frequency $f_x = 1/\lambda_x$ insted of wavenumber $k_x = 2\pi/\lambda_x$ Here $e^{-i(k_x x + k_y y)}$ or $e^{j2\pi(f_X x + f_Y y)}$ are the elementary components that the signal is made up of and $F(k_x,k_y)$ or $G(f_X, f_Y)$ is a complex function that containes information about the phase and amplitude of each elementary components.

Fourier transform theorems I

1. Linearity theorem: transform of sum is sum of transforms

$$\mathcal{F}\left\{\alpha g+\beta h\right\}=\alpha\mathcal{F}\left\{g\right\}+\beta\mathcal{F}\left\{h\right\}$$

2. Similarity theorem: sreatch of coordinates in space domain results in contraction of coordinates in frequency domain and a change in amplitude of the spectrum

If $\mathcal{F}\left\{g(x,y)\right\}=G\left(f_X,f_Y\right)$ then $\mathcal{F}\left\{g(ax,by)\right\}=\dfrac{1}{|ab|}G\left(\dfrac{f_X}{a},\dfrac{f_Y}{b}\right)$

3. Shift theorem: translation is space domain produces a linear phase shift in frequency domain.

If $\mathcal{F}\left\{g(x,y)\right\}=G\left(f_X,f_Y\right)$ then $\mathcal{F}\left\{g(x-a,y-b)\right\}=\underbrace{G\left(f_X,f_Y\right)}_{\text{Fourier transform is not affected}}\underbrace{e^{-j2\pi(f_Xa+f_Yb)}}_{\text{Phase shift}}$

Fourier transform theorems II

4. Rayleigh's enrgy theorem (parsewal's theorem): the sum (or integral) of the square of a function is equal to the sum (or integral) of the square of its transform.

If $\mathcal{F}\left\{g(x,y)\right\} = G\left(f_X, f_Y\right)$

Then $$\underbrace{\int\int_{-\infty}^{\infty} |\, g(x,y)\,|^2\, dxdy}_{\text{Energy contained in the waveform } g(x,y)} = \int\int_{-\infty}^{\infty} \underbrace{|\, G(f_X, f_Y)\,|^2}_{\text{energy density in frequency domain}} df_X\, df_Y$$

For the Fourier series the Parsewal's theorem takes the following form:

$$\frac{1}{L}\int_{-L}^{L}\left\{f(x)\right\}^2 dx = \frac{A_0^2}{2} + \sum_{m=1}^{\infty}\left(A_m^2 + B_m^2\right)$$

5. Convolution theorem:

If $\mathcal{F}\left\{g(x,y)\right\} = G\left(f_X, f_Y\right)$ and $\mathcal{F}\left\{h(x,y)\right\} = H\left(f_X, f_Y\right)$ then

$$\mathcal{F}\left\{\int\int_{-\infty}^{\infty} g(\xi,\eta)h(x-\xi, y-\eta)\,d\xi\, d\eta\right\} = G\left(f_X, f_Y\right)H\left(f_X, f_Y\right)$$

Convolution in space domain $\Leftrightarrow$ multiplication in frequency domain.

Fourier transform theorems III

6. Cross-correlation is a measure of similarity of two waveforms as a function of a time-lag applied to one of them. It is commonly used to search a long duration signal for a shorter, known feature. It also has applications in pattern recognition, single particle analysis, electron tomographic averaging, cryptanalysis, and neurophysiology.

$\mathcal{F}\{g(x,y)\} = G(f_X, f_Y)$ & $\mathcal{F}\{f(x,y)\} = F(f_X, f_Y)$ then cross-correlation of f, g is

$$h(x,y) = f * g = \int\int_{-\infty}^{\infty} f(\xi,\eta) g^*(\xi - x, \eta - y) d\xi d\eta$$

Autocorrelation is the cross-correlation of a function with itself. If $\mathcal{F}\{g(x,y)\} = G(f_X, f_Y)$

then
$$\begin{cases} \mathcal{F}\left\{ \int\int_{-\infty}^{\infty} g(\xi,\eta) g^*(\xi - x, \eta - y) d\xi d\eta \right\} = \left| G(f_X, f_Y) \right|^2 \\ \mathcal{F}\left\{ g(x,y)^2 \right\} = \int\int_{-\infty}^{\infty} G(\xi,\eta) G^*(\xi - f_X, \eta - f_Y) d\xi d\eta \end{cases}$$
The theorem is a special case of the convolution theorem in which we convolve $g(x,y)$ with $g^*(-x,-y)$

7. Fourier integral theorem: at each point of continuity of g,

$$\mathcal{F}\mathcal{F}^{-1}\{g(x,y)\} = \mathcal{F}^{-1}\mathcal{F}\{g(x,y)\} = g(x,y)$$

The two successive transforms yeild $\mathcal{F}\left\{ \mathcal{F}\{g(x,y)\}\right\} = ag(-x,-y)$

the successive transformation and inverse transformation of a function are not exactly the same although for even functions they differ within a constant.

8. Fourier transform of a separable function can be written as:

$\mathcal{F}\{ g(x,y)\} = \mathcal{F}_X\{ g_X(x)\} \mathcal{F}_Y\{ g_Y(y)\}$ if $g(x,y) = g_X(x) g_Y(y)$

Exercise

4.4) Fourier transform of the square pulse $f\left(x\right)=\begin{cases} E_f & -L/2\le x\le L/2 \\ 0 & |x|>L/2 \end{cases}$ is

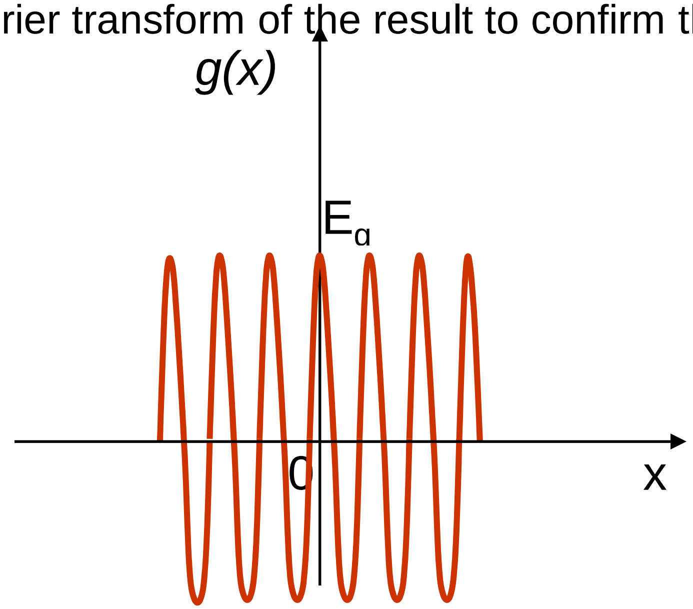
$\mathcal{F}\left\{f\left(x\right)\right\}=E_f L \sin c\dfrac{kL}{2}$. Use the Fourier transform theorems to find the Fourier tansform of the following pulses. In each case mention the theorem used.

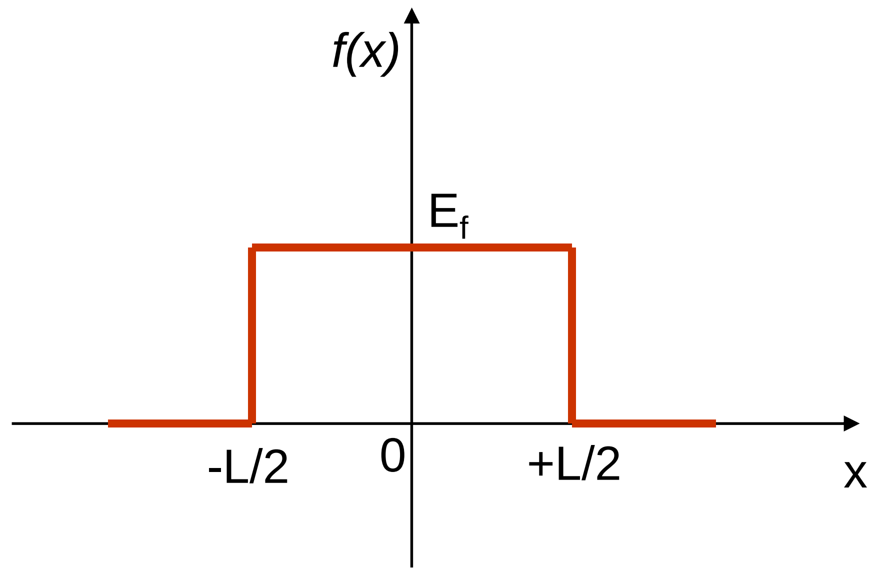
a) $f_a\left(x\right)=\begin{cases} E_f & -L\le x\le L \\ 0 & |x|>L \end{cases}$ b) $f_b\left(x\right)=\begin{cases} E_f & -aL\le x\le aL \\ 0 & |x|>aL \end{cases}$ c) $f_c\left(x\right)=\begin{cases} E_f & -L/b\le x\le L/b \\ 0 & |x|>L/b \end{cases}$

d) $f_d\left(x\right)=\begin{cases} E_f & -\alpha L\le x\le \beta L \\ 0 & x<-\alpha L,\ x>\beta L \end{cases}$ e) $f_d\left(x\right)=\begin{cases} E_f & 5L\le x\le 10L \\ 0 & x<5L,\ x>10L \end{cases}$

f) If $g(x)=E_g\cos k_g x$ find $\mathcal{F}\left\{g(x)\right\}$ then $\mathcal{F}\left\{f(x)+g(x)\right\}$; $\mathcal{F}\left\{f^2(x)\right\}$; $\mathcal{F}\left\{g^2(x)\right\}$ $\mathcal{F}\left\{f(x)*g(x)\right\}$ where * is the sign for convolution.

g) Calculate $f(x)*g(x)$ directly without using any of the theorems. Then take the Fourier transform of the result to confirm the convolution theorem.





Delta function

The delta dunction is a Generalized Function that is defined as the limit of a class of delta sequences. It is also called "Dirac's delta function" or "impulse symbole".

A generalized function is generalization of the concept of a function and they are particularly useful in making discontinuous functions more like smooth functions.

A delta sequence is a sequence of strongly peaked functions for which

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} \delta_n(x) f(x)dx = f(0)$$

As n $\to \infty$ the sequences become delta functions.

A Fundamental property of the delta function:

I) $\int_{-\infty}^{\infty} f(x)\delta(x-\alpha)dx = f(\alpha)$ and in fact $\int_{\alpha-\varepsilon}^{\alpha+\varepsilon} f(x)\delta(x-\alpha)dx = f(\alpha)$

More identities with delta function

$II)\ \delta(x-\alpha)=0$ for $x\neq\alpha$ $\qquad III)\ \delta(\alpha x)=\frac{1}{|\alpha|}\delta(x)$

$$IV)\ \delta(x^2-\alpha^2)=\frac{1}{2|\alpha|}[\delta(x+\alpha)+\delta(x-\alpha)]$$

$V)\ \delta[g(x)]=\sum_i \frac{\delta(x-x_i)}{|g'(x_i)|}$ where x_i are the roots of g

$VI)\ x\delta'(x)=-1(x)$ or more generally $x^n\delta^{(n)}=(-1)^n n!\delta(x)$

where $\delta^{(n)}$(x) is nth derivative of the delta function with respect to x

$VII)\ \delta'(-x)=-\delta'(x)$

$VIII)\ \int_{-\infty}^{\infty} f(x)\delta'(x-\alpha)dx=-f'(\alpha)$ which is equivalent to convolution

$(\delta'* f)(\alpha)=\int_{-\infty}^{\infty}\delta'\left(\alpha-x\right)f(x)dx=-f'(\alpha)$??

$IX)\ \int_{-\infty}^{\infty}|\delta'(x)|dx=\infty \qquad X)x^2\delta'(x)=0 \qquad XI)\int_{-1}^{1}\delta\left(\frac{1}{x}\right)dx=0$

XII) $\int f(x)\delta(x-x_0)dx=f(x_0)$ sifting property

Delta function in higher dimensions

In two dimension:

$$\delta^2(x,y)=\begin{cases}0 & x^2+y^2\neq 0\\ \infty & x^2+y^2=0\end{cases}$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\delta^2(x,y)dxdy=1$$

$$\delta^2(ax,by)=\frac{1}{ab}\delta^2(x,y)$$

$$\delta^2(x,y)=\delta(x)\delta(y)$$

In polar coordinates:

$$\delta^2(r,\theta)=\frac{\delta(r)}{\pi|r|}$$

In three dimension:

$$\delta^3(x,y,z)=\begin{cases}0 & x^2+y^2+z^2\neq 0\\ \infty & x^2+y^2+z^2=0\end{cases}$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\delta^3(x,y,z)dxdydz=1$$

$$\delta^3(ax,by,cz)=\frac{1}{abc}\delta^3(x,y,z)$$

$$\delta^3(x,y,z)=\delta(x)\delta(y)\delta\left(z\right)$$

In cylinderical coordinates:

$$\delta^3(r,\theta,z)=\frac{\delta(r)\delta(z)}{\pi r}$$

In polar coordinates:

$$\delta^3(r,\theta,\phi)=\frac{\delta(r)}{2\pi r^2}$$

Some delta sequences*

If we take limit of any of these sequences and let n $\rightarrow \infty$, we will have a delta function.

$$1)\ \delta(x)=\lim_{n\to\infty}\begin{cases} n & -\frac{1}{2n}<x<\frac{1}{2n} \\ 0 & \left|\frac{1}{2n}\right|<x \end{cases} \quad \rightarrow \delta_n(x)=\begin{cases} n & -\frac{1}{2n}<x<\frac{1}{2n} \\ 0 & \left|\frac{1}{2n}\right|<x \end{cases}$$

$$2)\ \delta(x)=\lim_{n\to\infty}\frac{n}{\sqrt{\pi}}e^{-n^2x^2} \quad \rightarrow \delta_n(x)=\frac{n}{\sqrt{\pi}}e^{-n^2x^2}$$

$$3)\ \delta(x)=\lim_{n\to\infty}\frac{n}{\pi}\sin c(nx) \quad \rightarrow \delta_n(x)=\frac{n}{\pi}\sin c(nx)$$

$$4)\ \delta(x)=\lim_{n\to\infty}\frac{1}{nx}\frac{e^{inx}-e^{-inx}}{2i} \quad \rightarrow \delta_n(x)=\frac{1}{nx}\frac{e^{inx}-e^{-inx}}{2i}$$

$$5)\ \delta(x)=\lim_{n\to\infty}\frac{1}{2\pi ix}\left[e^{ixt}\right]_{-n}^{n} \quad \rightarrow \delta_n(x)=\frac{1}{2\pi ix}\left[e^{ixt}\right]_{-n}^{n}$$

$$6)\ \delta(x)=\lim_{n\to\infty}\frac{1}{2\pi}\int_{-n}^{n}e^{ixt}dt \quad \rightarrow \delta_n(x)=\frac{1}{2\pi}\int_{-n}^{n}e^{ixt}dt$$

$$7)\ \delta(x)=\lim_{n\to\infty}\frac{1}{2\pi}\frac{\sin\left[\left(n+\frac{1}{2}\right)x\right]}{\sin\left(\frac{1}{2}x\right)} \quad \rightarrow \delta_n(x)=\frac{1}{2\pi}\frac{\sin\left[\left(n+\frac{1}{2}\right)x\right]}{\sin\left(\frac{1}{2}x\right)}$$

*Arfken

Some delta functions*

1) $\delta(x) = \frac{1}{\pi} \lim_{\varepsilon \to 0} \frac{\varepsilon}{x^2 + \varepsilon^2};$

2) $\delta(x) = \lim_{\varepsilon \to 0} \varepsilon \left|x\right|^{\varepsilon - 1};$

3) $\delta(x) = \lim_{\varepsilon \to 0^+} \frac{1}{2\sqrt{\pi\varepsilon}} e^{-x^2/4\varepsilon};$

4) $\delta(x) = \lim_{\varepsilon \to 0} \frac{1}{\pi x} \sin\left(\frac{x}{\varepsilon}\right);$

5) $\delta(x) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} Ai\left(\frac{x}{\varepsilon}\right);$ Ai is the Airy Function $Ai(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i(xt + t^3/3)} dt$

6) $\delta(x) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} J_{1/\varepsilon}\left(\frac{x+1}{\varepsilon}\right);$ $J_n(x)$ is a Bessel function of the first kind

7) $\delta(x) = \lim_{\varepsilon ® 0} \left|\frac{1}{\varepsilon} e^{-x^2/\varepsilon} L_n\left(\frac{2x}{\varepsilon}\right)\right|;$ L_n is a Laguerre polynomial of an arbitrary

positive integer order and $L_n(x) = \frac{1}{2\pi i} \oint \frac{e^{-xt/(1-t)}}{(1-t)t^{n+1}} dt$ contur encloses the

origin and traversed in a counterclockwise direction.

*Arfken

The delta function as a Fourier transform

The delta function as a Fourier transform is:

$$\delta(x) = \mathcal{F}_k\left[1\right] = \int_{-\infty}^{\infty} e^{jkx} dk$$

$$\mathcal{F}_x^{-1}[\delta(x)] = \int_{-\infty}^{\infty} \delta(x) e^{-jkx} dx = 1$$

Fourier transform of the delta function is

$$\mathcal{F}_x\left[\delta(x - x_0)\right] = \int_{-\infty}^{\infty} e^{jkx} \delta(x - x_0) dx = e^{jkx_0}$$

Fourier transform of the Gaussian Function

Example: $f(x) = Ce^{-ax^2}$ is profile of a Gaussian pulse at $t = 0,$ where a is a constant and $C = \sqrt{a/\pi}$. Prove that the Fourier transform of $f(x)$ is a Gaussian function. Then show that the product of the widths of the function and its Fourier transfom is a constant.

What is the value of the constant if we choose to measure the width at $e^{-1/2}$ of their maxima?

Ans) $F(K) = \int_{-\infty}^{\infty} (Ce^{-ax^2}) e^{ikx} dx = e^{-k^2/4a}, \ \sigma_x = \dfrac{1}{\sqrt{2a}}, \ \sigma_k = \sqrt{2a}$

$$-ax^2 + ikx = -\Bigg(\underbrace{\sqrt{a}x - ik/2\sqrt{a}}_{\text{change variable to } \beta}\Bigg)^2 - k^2/4a = -\beta^2 - \frac{k^2}{4a}, \ dx = \frac{d\beta}{\sqrt{a}}$$

$$F(K) = \sqrt{\frac{a}{\pi}} \frac{1}{\sqrt{a}} e^{-k^2/4a} \int_{-\infty}^{\infty} e^{-\beta^2} d\beta = \frac{1}{\sqrt{\pi}} e^{-k^2/4a} \sqrt{\pi} \to \boxed{F(K) = e^{-k^2/4a}}$$

The Fourier transform of a Gaussian is a Gaussian function with a different coefficients.

We measure the width of $f(x)$ and $F(K)$ at $1/\sqrt{e} = 0.607$ of their maximum and call the corresponding x and k the standard deviation $(\sigma_x$ and $\sigma_k)$ of the function and its Fourier transform. $\sigma_k{}^2/4a = 1/2 \to \sigma_k = \sqrt{2a}$; $a\sigma_x{}^2 = 1/2 \to \sigma_x = 1/\sqrt{2a}$; $\sigma_x \sigma_k = 1$

Product of the pulsewidth and it spatial frequency bandwidth is constant.

Applications of the Gaussian function: wave packet of individual photons,cross-sectional irradiance distribution of a laser beam in TEM_{00} mode can be represented with Gaussian profile.

Example of Generalized Fourier transform

Fourier transform of the Dicac delta function:

$$\delta(x,y) = \lim_{N \to 0} N^2 e^{-N^2 \pi (x^2+y^2)}$$

$$\mathcal{F}\left\{g(x,y)\right\} = \mathcal{F}\left\{N^2 e^{-N^2 \pi (x^2+y^2)}\right\} = \int\int_{-\infty}^{\infty} N^2 e^{-N^2 \pi (x^2+y^2)} e^{-j2\pi (f_x x + f_y y)} dxdy$$

Exercise

4.5)

a) Find $\delta(x^2 + x - 2)$ using identity V

b) Find the Fourier series expansion of $\delta(x - \alpha)$