Fourier Analysis

Anharmonic periodic waves

Fourier series

Fourier Optics

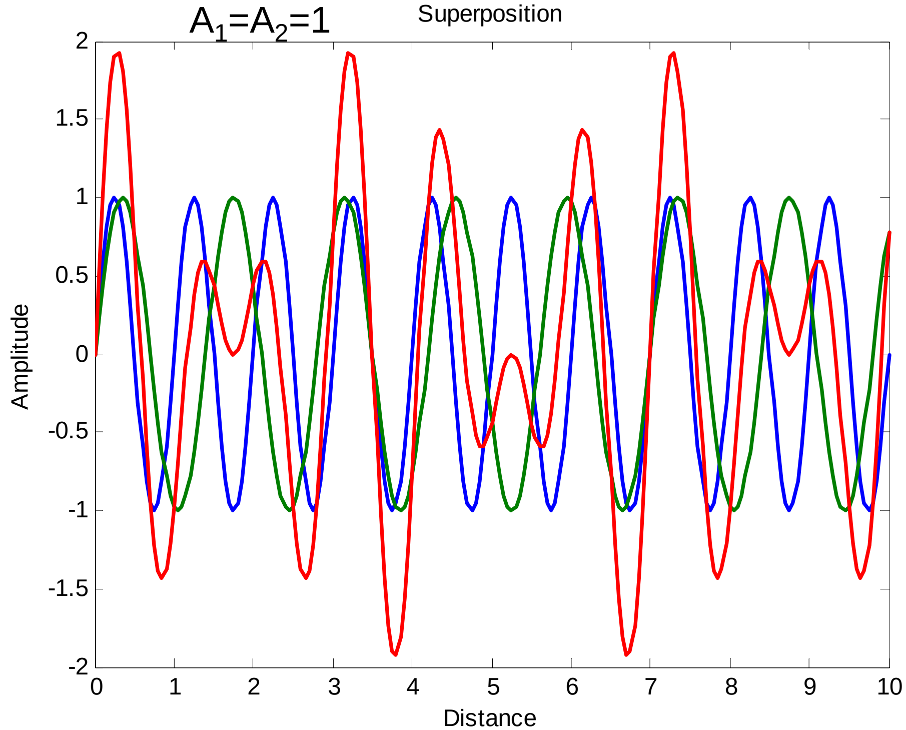
- Study of light with the mathematical techniques and insight of communications theory (1950s) bound with the mathematical formalism of Fourier analysis. Some of the results are:
  - Image formation and evolution
  - Transfer functions
  - Spatial filtering
- Fourier Theorem (1768 Jean Baptist Joseph, Baron de Fourier)
  - A function of *f(x),* having a spatial period of λ, can be synthesized by a sum of harmonic functions whose wavelengths are integral sub-multiples of λ (that is λ, λ /2, λ /3, …)

$$f(x) = C_0 + C_1 \cos\left(\frac{2\pi}{\lambda}x + \varepsilon_1\right) + C_2 \cos\left(\frac{2\pi}{\lambda/2}x + \varepsilon_2\right) + C_3 \cos\left(\frac{2\pi}{\lambda/3}x + \varepsilon_3\right) + \ldots$$

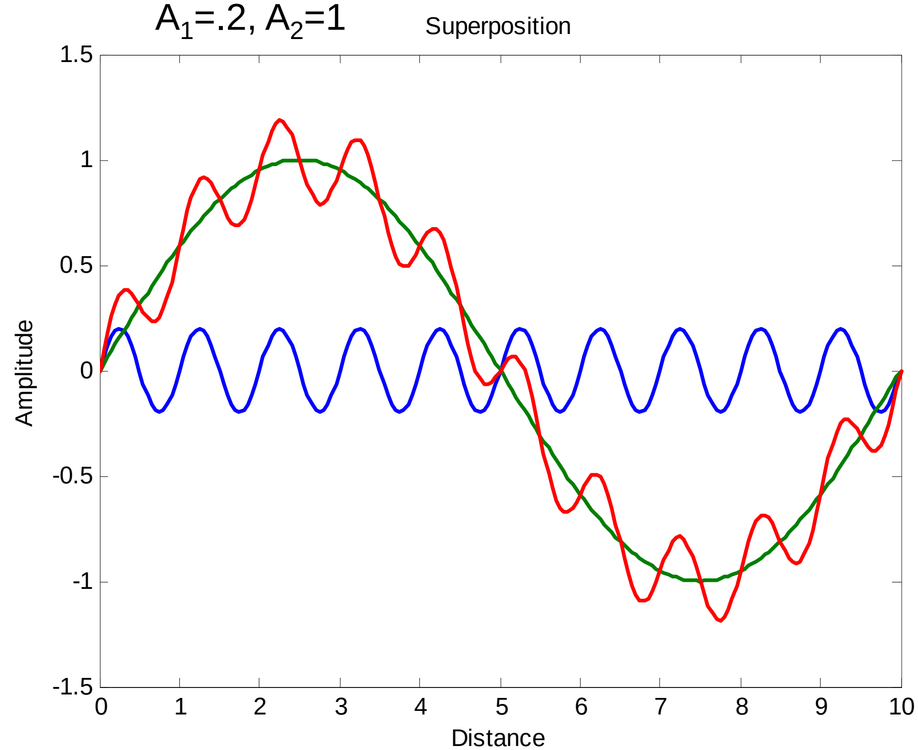
Anharmonic periodic waves: periodic but not sinusoidal

- How we can construct any real wave out of appropriately chosen harmonic waves?
- What is the spatial period of superposition of two waves?

$\lambda_1=1$
$\lambda_2=1.4$
$N_1\lambda_1=N_2\lambda_2$
$\lambda=7\ \lambda_1=5\ \lambda_2$
$A_1=A_2=1$

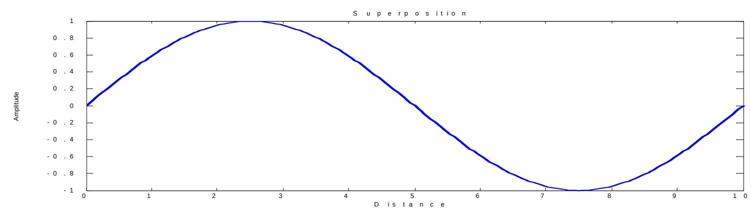


$\lambda_1=1$
$\lambda_2=10$
$N_1\lambda_1=N_2\lambda_2$
$\lambda=10\lambda_1=1\ \lambda_2$
$A_1=.2,\ A_2=1$

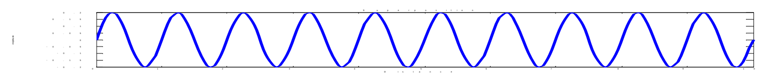


Frequency spectrum

Ingredients

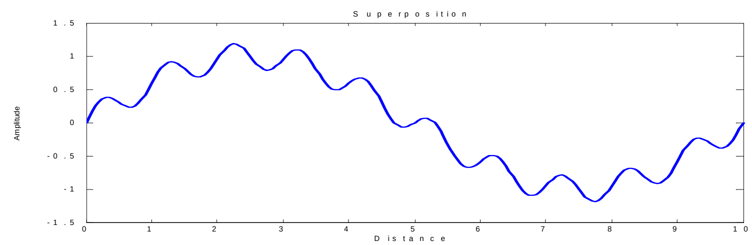


ν_1

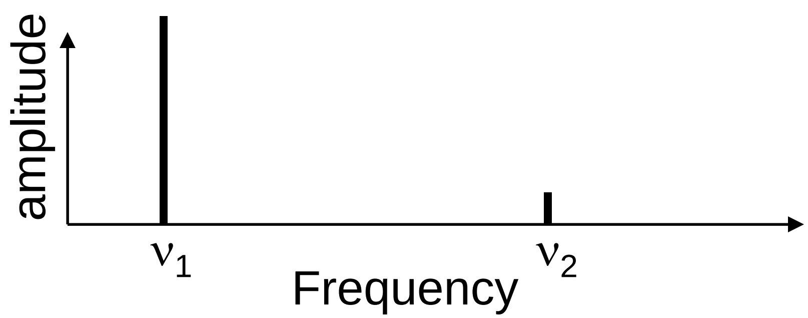


ν_2

Composite wave



Frequency spectrum



Mathematical form of the Fourier series representation

$$f(x) = C_0 + C_1 \cos(\frac{2\pi}{\lambda} x + \varepsilon_1) + C_2 \cos(\frac{2\pi}{\lambda/2} x + \varepsilon_2) + C_3 \cos(\frac{2\pi}{\lambda/3} x + \varepsilon_3) + ...$$

$f(x)$ is a periodic function

The more terms we have, the better an arbitrary anharmonic function is represented.

If we have infinite number of terms in the Fourier series, $f(x)$ can fit any arbitrary anharmonic function with any period. Using

$C_m \cos\left(mkx + \varepsilon_m\right) = A_m \cos mkx + B_m \sin mkx$ the $f(x)$ can also be written as:

$$f(x) = \frac{A_0}{2} + \sum_{m=1}^{\infty} A_m \cos mkx + \sum_{m=1}^{\infty} B_m \sin mkx$$

Where $\underbrace{A_m = C_m \cos \varepsilon_m; \quad B_m = -C_m \sin \varepsilon_m;}_{\text{Together they build the AC components.}}$ $\quad \underbrace{A_0 / 2 = C_0}_{\text{DCcomponent of the wave anharmonic function}}$

Fourier Analysis : finding A_0, A_m, B_m for a specific periodic function.

Fourier Analysis

$$f(x) = \frac{A_0}{2} + \sum_{m=1}^{\infty} A_m \cos mkx + \sum_{m=1}^{\infty} B_m \sin mkx$$

Using the orthognality of $\sin$ and $\cos$ functions we can find $A_0,\ A_m,\ B_m$

$$A_0 = \frac{2}{\lambda}\int_0^{\lambda} f(x)dx; \quad A_m = \frac{2}{\lambda}\int_0^{\lambda} f(x)\cos(mkx)dx; \quad B_m = \frac{2}{\lambda}\int_0^{\lambda} f(x)\sin(mkx)dx$$

For <u>even functions</u> or symmetric functions about x,

$f(-x) = f(x) \ \rightarrow \ B_m = 0$ for all m (can't contain sin components)

For <u>odd functions</u> or anti-symmetric functions about x,

$f(-x) = -f(x) \rightarrow A_m = 0$ for all m (can't contain cos components)

$\frac{A_0}{2}$ is the mean value of the $f(x)$ or the DC component of the waves

<u>Advantage of Fourier expansion:</u> it is possible to expand discontinious functions using Fourier expansion that Taylor or other expansions can not represent them. <u>Because there is no derivative</u>.

Change of intervals and complex Fourier series

In general for a piecewise regular function with a spatial period of λ

we have:

$$f(x) = \frac{A_0}{2} + \sum_{m=1}^{\infty} A_m \cos\frac{m\pi x}{\lambda/2} + \sum_{m=1}^{\infty} B_m \sin\frac{m\pi x}{\lambda/2}$$

$$A_m = \frac{2}{\lambda}\int_{x_0}^{x_0+\lambda} f(x)\cos\frac{m\pi x}{\lambda/2}dx;\ B_m = \frac{2}{\lambda}\int_{x_0}^{x_0+\lambda} f(x)\sin\frac{m\pi x}{\lambda/2}dx;\ A_0 = \frac{2}{\lambda}\int_{x_0}^{x_0+\lambda} f(x)dx$$

Integration can be over any interval of $(x_0, x_0+\lambda)$. One may use $k = 2\pi/\lambda$

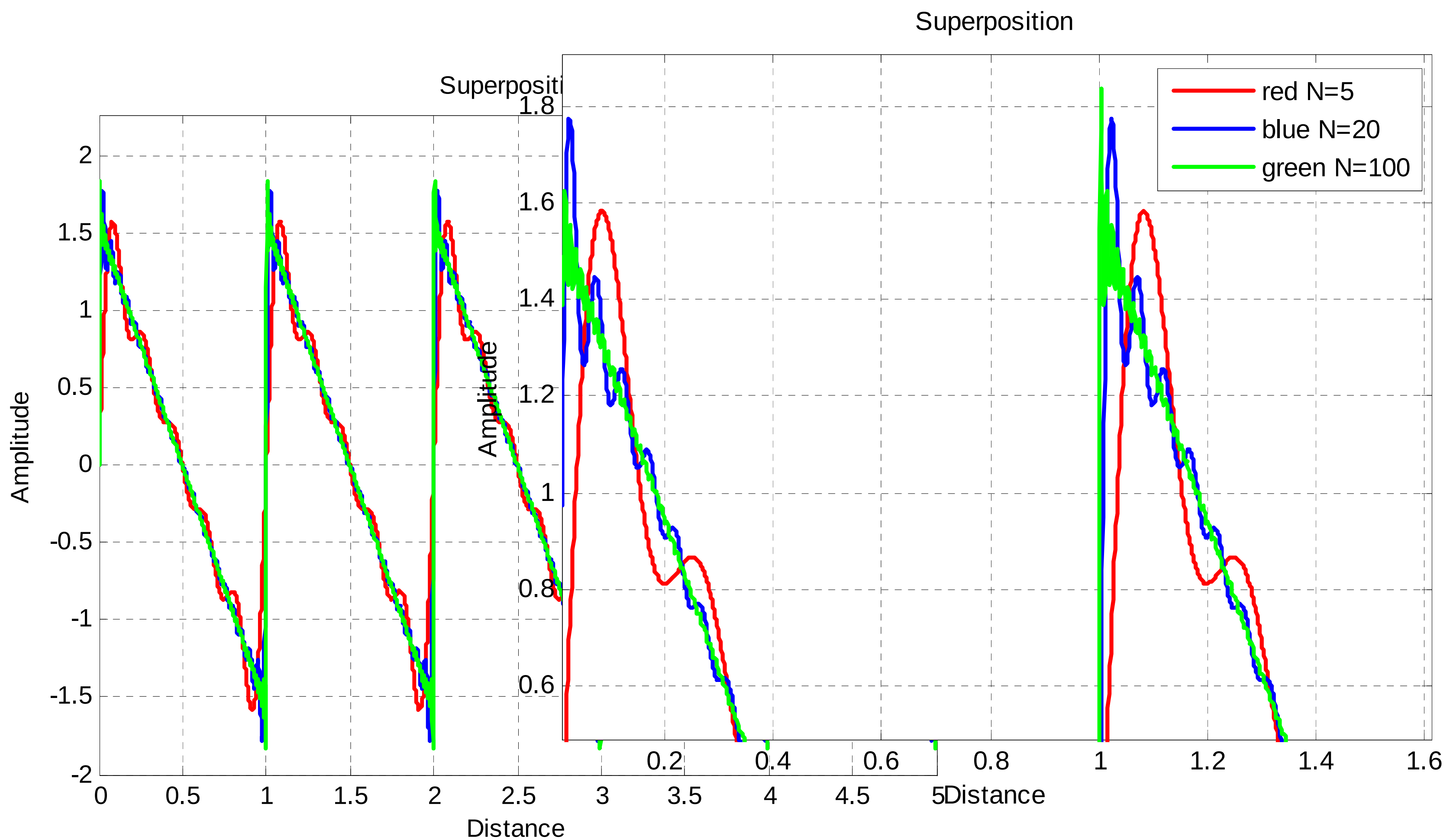
Complex Fourier series: $f(x) = \sum_{m=-\infty}^{\infty} C_m e^{imkx}$

$$C_m = \frac{1}{2}(A_m - iB_m);\quad C_{-m} = \frac{1}{2}(A_m + iB_m);\quad C_0 = \frac{1}{2}A_0 \quad \text{or}$$

$$C_m = \frac{1}{\lambda}\int_{x_0}^{x_0+\lambda} f(x)e^{-i\frac{m2\pi x}{\lambda}}dx = \frac{1}{\lambda}\int_{x_0}^{x_0+\lambda} f(x)e^{-imkx}dx$$

Superposition of many waves
Accuracy and number of Fourier components

Plot shows superposition of N harmonic waves with wavelengths and amplitudes of each wave one unit less than the previous one.



Exercise

3.1) Compute the Fourier series that corresponds to a square wave.

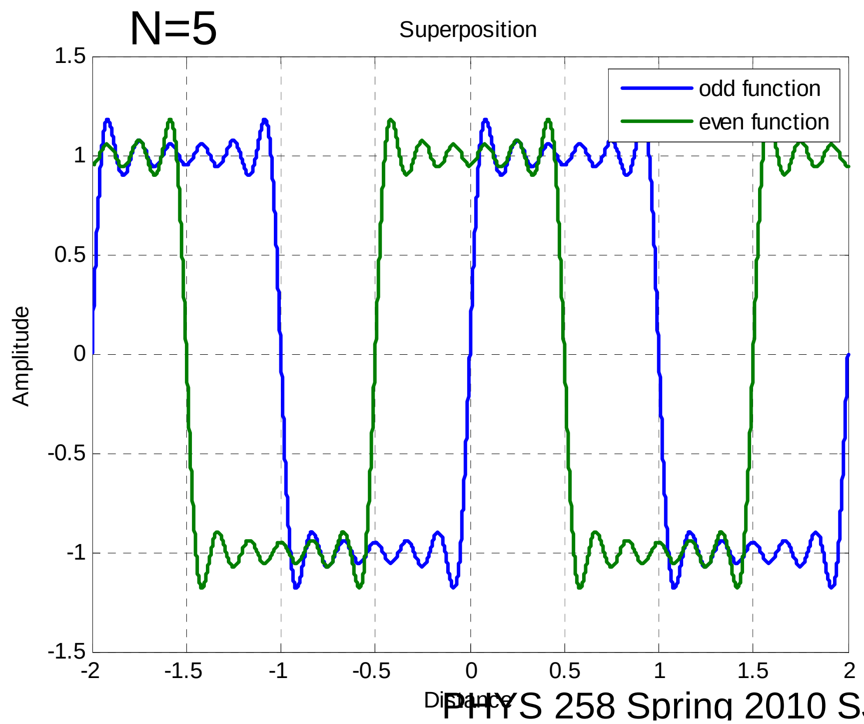
Amplitude of the wave is 1 and $f(0) = +1$. $f_{even}(x) = \begin{cases} +1 \text{ when } -\lambda/4 < x \leq \lambda/4 \\ -1 \text{ when } \lambda/4 < x \leq 3\lambda/4 \end{cases}$

Plot your results for both odd and even function for i=5 and i=100 where i is the number of fourier components superimposed .

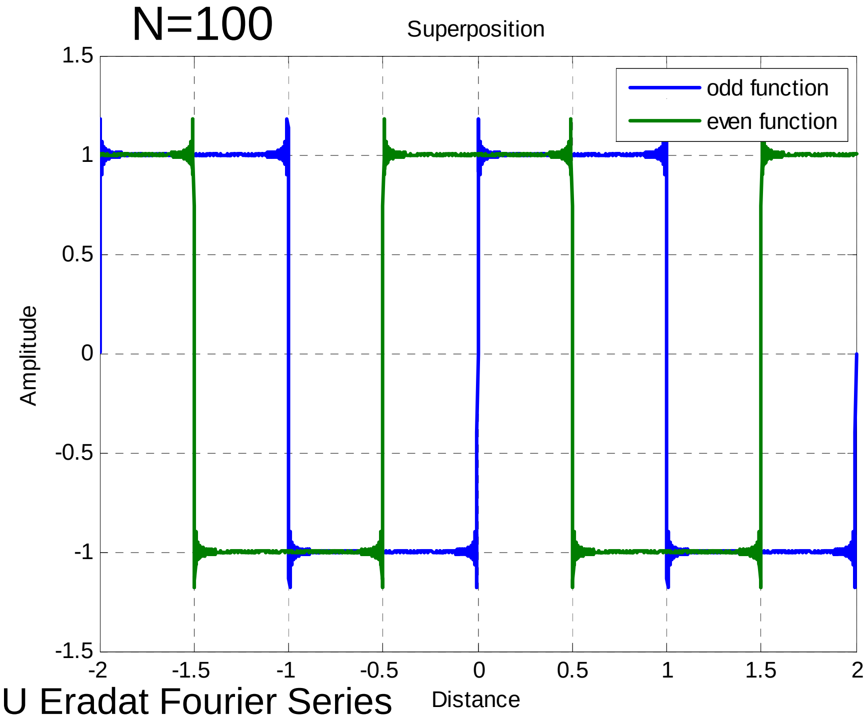
The odd function of the same wave is: $f_{odd}(x) = \begin{cases} +1 \text{ when } 0 < x \leq \lambda/2 \\ -1 \text{ when } \lambda/2 < x \leq \lambda \end{cases}$

The Fourier series for this function can be found in Hecht.

N=5



N=100



Exercise

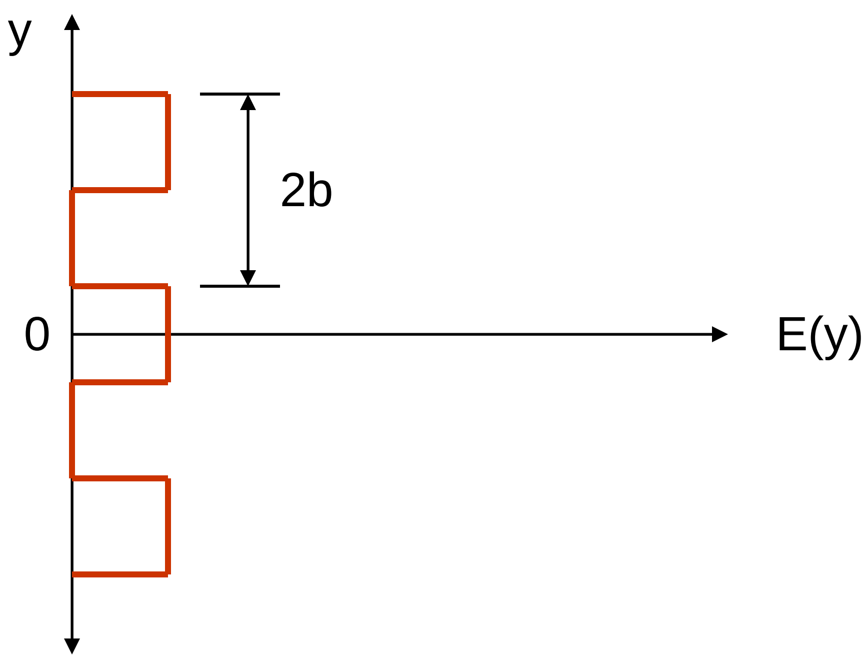
3.2) Find the Fourier series that represents the following function. First plot the function to see its shape, odd and evenness.

$$f(x) = \begin{cases} \frac{1}{2}(n-x) & 0<x\leq\pi \\ -\frac{1}{2}(n+x) & -\pi\leq x<0 \end{cases}$$

Answr: $\sum_{m=1}^{\infty}\frac{\sin mx}{m}$

Exercise: A real application

3.3) A plane wave of amplitude E_0 impinging normally on a large horizontal Ronchi ruling that is a grating formed of alternatively transparent and opaque stripes, each of width b. The emerging electric field over the screen or aperture is a step function. Compute its Fourier series representation, assuming it to have effectively infinite extend.



Time domain

For any anharmonic periodic prograssive disturbance (wave) we can write

$$f(x \pm Vt) = \frac{A_0}{2} + \sum_{m=1}^{\infty} A_m \cos mk(x \pm Vt) + \sum_{m=1}^{\infty} B_m \sin mk(x \pm Vt)$$

or

$$f(x \pm Vt) = \sum_{m=1}^{\infty} C_m \cos[mk(x \pm Vt) + \varepsilon_m]$$

Nonperiodic waves

- In optics and quantum mechanics all real waves are pulses.
- In order to generate a pulse out of harmonic functions that have a certain width and shape, we need to know
  - what frequency elements to add
  - How much of each frequency element to add
- That is finding the frequency spectrum of the pulses.

Addition of waves: different frequencies

To generate beats we added two frequencies ω_1 and ω_2

$$E = 2E_{01}\cos\left[k_m x - \omega_m t\right] \times \cos\left[\bar{k}x - \bar{\omega}t\right]$$

the carrier frequency is average of the added frequencies $\bar{\omega} = \frac{1}{2}\left(\omega_1 + \omega_2\right)$

1) If we add more frequency elements symmetrically around $\bar{\omega}$, then $\bar{\omega}$ will not change.

2) If we reduce the spacing between the frequency elements, then frequency of modulation $\omega_m = \frac{1}{2}\left(\omega_1 - \omega_2\right)$ will decrease. This is the frequency of the envelope meaning the wavelength of the beats will increase and beats will have more separation.

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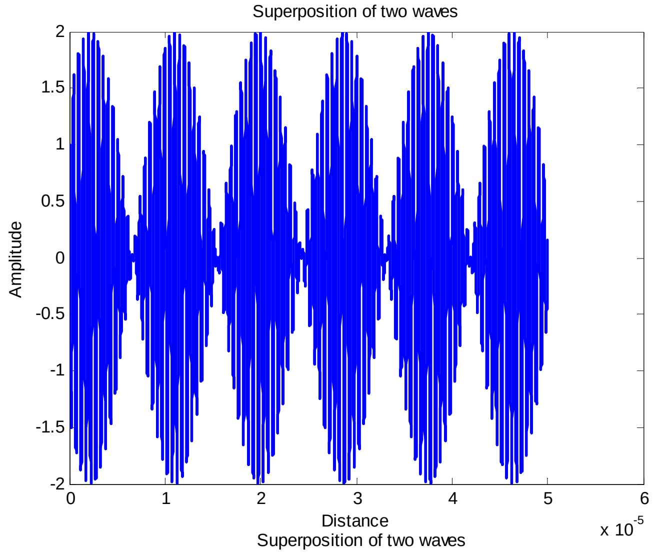
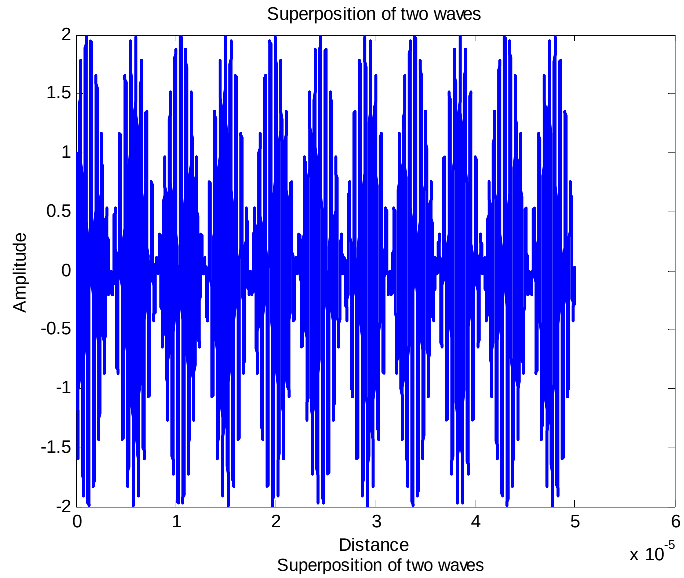
3) When number of frequency elements go to infinity, we will have a single pulse. The λ_m goes to infinity.

Beats with different frequency intervals

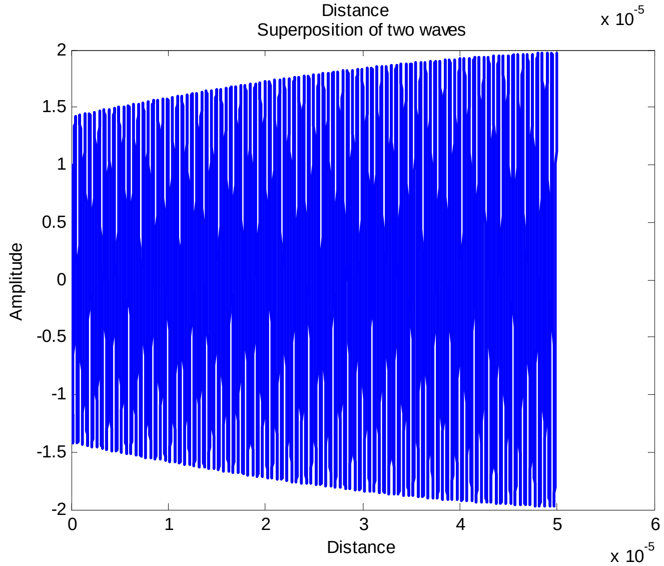
$\Delta\lambda$=30

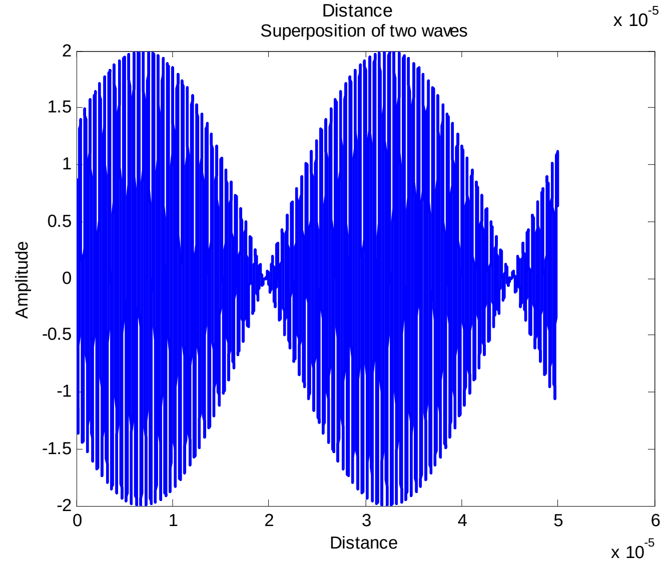




$\Delta\lambda$=60

$\Delta\lambda$=1





$\Delta\lambda$=1