Diffraction theory

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Diffraction

- Diffraction is a phenomena in the realm of physical optics. Applicable to all waves such as acoustic and EM waves.
- Diffraction is a limiting factor on data processing and imaging system performance. So we need to understand it to design better systems.
- Diffraction should not be mistaken with
  - Refraction: change of direction of propagation of light due to a change in index of refraction of the environment
  - Penumbra: finite extend of a source causes the light transmitted from an aperture to spread away from it. There is no bending of light involved in Penumbra effect.
- Diffraction (Sommerfeld): any deviation of light rays from rectilinear that <u>cannot be interpreted</u> as reflection or refraction.
- Diffraction is caused by <u>confinement of the lateral extend of a wave</u> (obstruction of the wavefront) and its effects are most pronounced when <u>size of the confinement is comparable to the wavelength</u> of the light.

History of diffraction theory

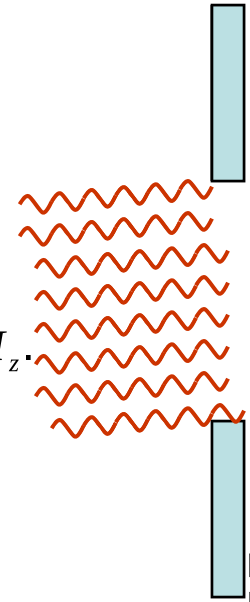
- 1665 Grimaldi reported diffraction for the first time.
- 1678 Huygens attempted to explain the phenomenon
  - Each point on the wavefront of a disturbance is considered to be a new source of a "secondary" spherical disturbance. Then the wavefront at later instances can be found by constructing the "envelope" of the secondary wavelet.
- 1700s Progress on wave theory was suppressed by the fact that Newton favored the corpuscular theory of light (geometrical optics).
- 1804 Thomas Young introduced the concept of interference to the wave theory of light (production of darkness from light).
- 1818 Augustin Jean Fresnel used the wavelets from Huygens theory and Young's interference theory letting the wavelets interfere mutually to calculate distribution of light in diffraction patterns with excellent accuracy.
- 1860 Maxwell identified light as electromagnetic field.
- 1882 Gustav Kirchhoff put the Fresnel and Maxwell's ideas together. He made two assumptions about the boundary values of the light incident on surface of an obstacle that were not absolutely correct but an approximation and constructed a theory that exhibited excellent agreement with experimental results. He concluded
  - The amplitudes and phases ascribed to the secondary sources of Huygens wavelets are logical consequences of the wave nature of the light.
- 1892 Poincare; 1894 Sommerfeld proved that the boundary values set by Kirchhoff are inconsistent with one another. So Kirchhoff's formulation of Huygens-Fresnel principle is regarded as the first approximation although under most conditions it yields excellent results.
- 1896 Sommerfeld modified the Kirchhoff's theory using theory of Green's function. The result is *Rayleigh-Sommerfeld diffraction theory*.
- 1923 Kottler: first satisfactory generalization of the vectorial diffraction theory.

From vector to a scalar theory I

- In all of these theories light is treated as a scalar phenomenon.
- At boundaries the various components of the electric and magnetic fields are coupled through Maxwell equations and cannot be treated independently.
- We stay away from those situations when using scalar theory.
- Scalar theory yields correct values under two conditions:
  - The diffracting aperture must be large compared with a wavelength.
  - The diffracting fields must not be observed too close to the aperture.
- Our treatment is not good for some optical systems such as diffraction from
  - high-resolution gratings
  - Small pits on optical recording media
- Read Goodman 3.2 From a Vector to a Scalar Theory

From vector to a scalar theory II

For electromagnetic waves propagating in media with the following properties, an scalar wave equation is obeyed by all components of the field vectors $E_x, E_y, E_z, H_x, H_y, H_z$.



$$\nabla^2 u(P,t) - \frac{n^2}{c^2}\frac{\partial^2 u(P,t)}{\partial t^2} = 0$$

$u(P,t)$ is any of the scalar field components x, y, z at time t.

Depth of penetration in the media is few wavelengths. Not much effect on total wavefront passing through the aperture

Properties:

linear; if $u_1(P,t)$ and $u_2(P,t)$ are solutions to the wave euation, then $\alpha u_1(P,t) + \beta u_2(P,t)$ is a solution,

isotropic; properties are independent of direction of polarization of the wave,

homogeneous; permitivity is constant throughout the region of propagation,

nondispersive; permitivity is independent of frequency over the region of propagation,

nonmagnetic; magnetic permeability is equal to μ_0

At the boundaries, the above criteria are not met and coupling betwen electric and magnetic components of the EM wave happens.

We can use the scalar theory if the boundaries are small portion of the total area through which the wave is passing.

The Helmholtz equation

For a monochromatic wave $u(P,t)$ the scalar field can be written as

$u(P,t) = A(P)\cos[2\pi\nu t - \phi(P)] = \mathrm{Re}\{U(P)e^{-j2\pi\nu t}\}$ where

$U(p) = A(p)e^{j\phi(P)}$ space dependent part of the field

$e^{-j2\pi\nu t}$ time dependent part of the field

P can be any of the space coordinates $x,\ y,$ or z

Substituting the scalar field in scalar wave equation

$$\nabla^2 u(P,t) - \frac{n^2}{c^2}\frac{\partial^2 u(P,t)}{\partial t^2} = 0$$

$$\nabla^2 U(P)e^{-j2\pi\nu t} - \frac{n^2}{c^2}\frac{\partial^2 U(P)e^{-j2\pi\nu t}}{\partial t^2} = 0$$

$$\left(\nabla^2 U(P) - \frac{n^2(-j2\pi\nu)^2}{c^2}U(P)\right)e^{-j2\pi\nu t} = 0$$ with wavenumber $k = \sqrt{\frac{n^2(2\pi\nu)^2}{c^2}} = \frac{2\pi}{\lambda}$

$\boxed{(\nabla^2 + k^2)U(P) = 0}$ time-independent Helmholtz equation.

Complex amplitude of any monochromatic wave propagating in vacuum or in homogeneous dielectric media has to obey the Helmholtz equation.

Gauss's Theorem

Gauss's theorem:

connecting surface integral and volume integral of a vector

$$\overbrace{\int_V \underbrace{\nabla \bullet \mathbf{U}}_{\substack{\text{net outflow}\\ \text{of flux per}\\ \text{unit volume}}} \, dV}^{\substack{\text{total outflow of flux}\\ \text{from the volume } V}} = \underbrace{\int_S \mathbf{U} \bullet d\mathbf{s}}_{\substack{\text{total outflow of flux}\\ \text{from the surface } S}}$$

Green's theorem is a corollary of Gauss's theorem

Green's Theorem: a mathematical tool

If U and G are two scalar functions

$$\nabla \bullet (U \nabla G) = U \nabla \bullet \nabla G + \left(\nabla U\right) \bullet \left(\nabla G\right) \quad -$$

$$\nabla \bullet (G \nabla U) = G \nabla \bullet \nabla U + \left(\nabla G\right) \bullet \left(\nabla U\right)$$

$$\nabla \bullet (U \nabla G - G \nabla U) = U \nabla^2 G - G \nabla^2 U$$

Assuming $U,\ G,$ their first and second derivatives are continuous over the volume V and on the surface S enclosing the V

$$\iiint_V \nabla \bullet (U \nabla G - G \nabla U) dV = \iiint_V (U \nabla^2 G - G \nabla^2 U) dV$$

Using the Gauss's theorem convert the volume integral on LHS

$$\iint_S (U \nabla G - G \nabla U) \bullet d\mathbf{s} = \iiint_V (U \nabla^2 G - G \nabla^2 U) dV$$

If we take the gradiant in the outward normal direction $\mathbf{n}$ the LHS can be written in a scalar form since $\mathbf{n} \parallel \mathbf{s}$ at every point.

$$\iint_S \left(U \frac{\partial G}{\partial n} - G \frac{\partial U}{\partial n}\right) ds = \iiint_V (U \nabla^2 G - G \nabla^2 U) dV \quad \leftarrow \text{Green's theorem}$$

Physical meaning of the Green's function I

Imagine an inhomogeneous linear differential euation

$$a_2 \frac{d^2U(x)}{dx^2} + a_1 \frac{dU(x)}{dx} + a_0 U(x) = V(x)$$

$V(x)$ is a driving force

$U(x)$ is the solution for a known set of boundary conditions (BC)

$G(x)$ is a solution to the equation with $V(x) \to \delta(x - x')$ the impulse driving force and the same BCs.

$G(x)$ is an impulse response and we can expand $U(x)$ in terms of Gs

$$U(x) = \int V(x')G(x - x')dx'$$

$G(x)$ known as the Green's function of the problem.

$G(x)$ may be regarded as an auxiliary function chosen cleverly to solve our problem.

Physical meaning of the Green's function II

Imagine an oscillator

$$a_2 \frac{d^2U(x)}{dx^2} + a_0U(x) = V(x)$$

$V(x)$ is a driving force

$U(x)$ is the solution for a known set of boundary conditions (BC)

$G(x)$ is impulse response

$$U(x) = \int_{x'} V(x')G(x - x')dx'$$

The solution is convolution of the driving force with the impulse respons of the system.

In case of diffraction application of Green's theorem will yeild different variations of the diffraction theory based on the choice of Green's function.

Application of Green’s Theorem in scalar diffraction theory

Goal: calculation of the complex disterbance U at an observation point in space, P_0, using Green's Theorem.

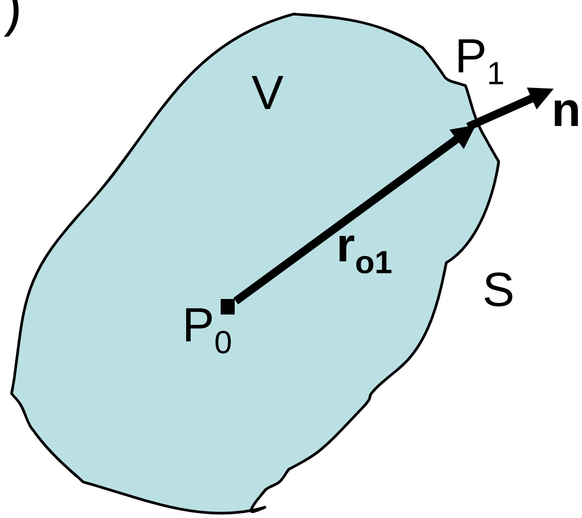
$$\iint_S \left(U\frac{\partial G}{\partial n} - G\frac{\partial G}{\partial n}\right)ds = \iiint_V (U\nabla^2 G - G\nabla^2 U)dV$$

Green's theorem is the prime foundation of the scalar diffraction theory.

To apply it to the diffraction problem we need to have a proper choice of

1) an auxiliary function G (Green's function)

2) a close surface S



P_0 is an arbitrary point of observation

P_1 is an arbitrary point on the surface S

We want solution of the wave equation $U(P_0)$ at P_0 in terms of the value of the solution and its derivatives on the surface S.

The integral theorem of Helmholtz & Kirchhoff

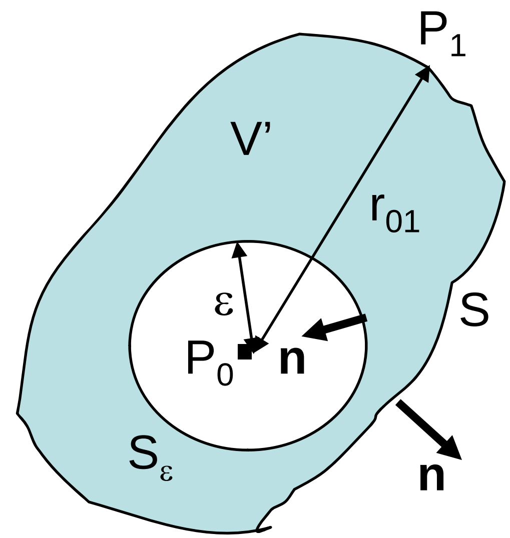
1) Choice of Green's function: a unit-amplitude spherical wave expanding about point P_0 (impulse). G has to be a solution of the wave equation. At P_1: $G(P_1) = \frac{e^{jkr_{01}}}{r_{01}}$

2) Treating the discontinuity at P_0 by isolating it with S_ε

3) New surface & volume: $S' = S + S_\varepsilon; \;\; V' = V - \frac{4}{3}\pi\varepsilon^3$

4) Use Green's theorm with Green's function of

$\boxed{G(P_1) = \frac{e^{jkr_{01}}}{r_{01}}}$ and Helmholtz equation $(\nabla^2 + k^2)U = 0$



Note both G and U are the solutions of the same wave euation. After all G is the impulse response and U is the disturbance. We want to find $U(P_0)$ or field after the aperture.

At the linit of $\varepsilon \to 0$ we get (follow from Goodman page 41):

$$\boxed{U\left(P_0\right) = \frac{1}{4\pi}\iint_S \left\{ \frac{\partial U}{\partial n}\frac{e^{jkr_{01}}}{r_{01}} \;\; \text{-} U\frac{\partial}{\partial n}\frac{e^{jkr_{01}}}{r_{01}} \right\} ds}$$

This result is known as the integral thorem of Helmholtz and Kirchhoff. It has important role in development of the scaler theory of diffraction. $U\left(P_0\right)$, the field at point P_0 is expressed in terms of the "boundary values" of the wave on any closed surface surrounding that point.

Fresnel-Kirchhoff diffraction formula I

Problem: diffraction of light by an aperture in an infinite opaque screen.

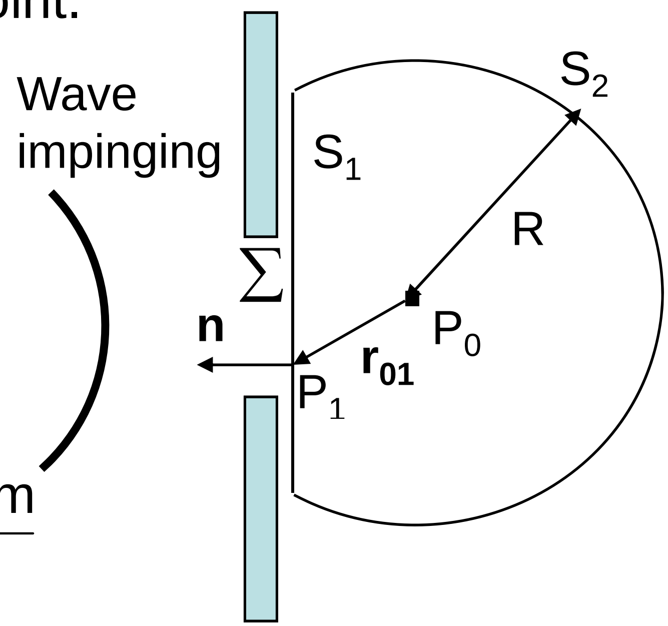
The field U at P_0 behind the aperture is to be calculated.

r_{01} is the distance from aperture to observation point.

Assumptions: $r_{01} >> \lambda$ and $k >> 1/r_{01}$

Choice of S: a plane surface plus a spherical cap $S = S_1 + S_2$

Choice of Green's function: $G = \dfrac{e^{jkr_{01}}}{r_{01}}$



We apply the Helmholtz-Kirchhoff integral theorem

$$U\left(P_0\right) = \frac{1}{4\pi} \iint_{S_1+S_2} \left(G\frac{\partial U}{\partial n} - U\frac{\partial G}{\partial n} \right) ds$$ to find $U(P_0)$

Somefeld radiation condition: if the disturbance U vanishes at least as fast as a diverging spherical wave then:

as $R \to \infty$ contribution of S_2 to $U(P_0)$ vanishes.

Kirchhoff's Boundary conditions

The screen is opaque and the aperture is shown by Σ

1) Across the surface Σ, the field and its derivatives are exactly the same as they would be in the absence of the screen.

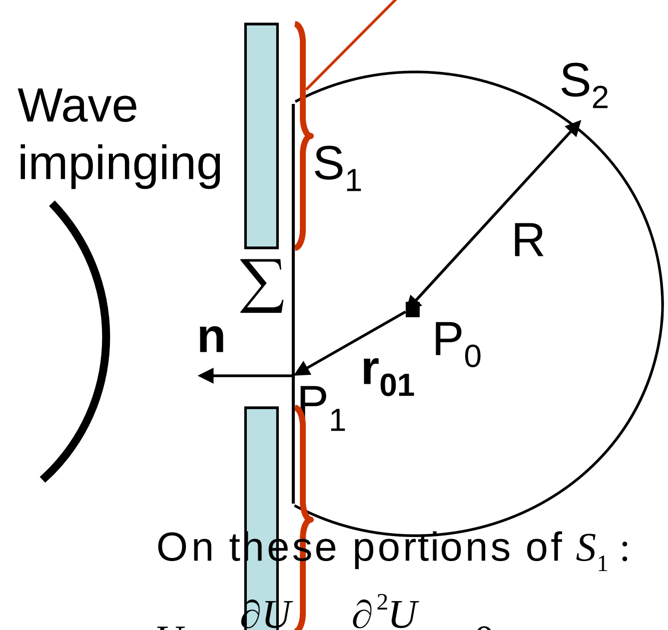
2) Over the portion of S_1 that lies in the geometrical shadow of the screen the filed and its derivative are identically zero.

$$U\left(P_0\right)=\frac{1}{4\pi}\iint_{\Sigma}\left(G\frac{\partial U}{\partial n}-U\frac{\partial G}{\partial n}\right)ds$$

Condition 1 is not exactly true since close to the boundaries the field is disturbed.

Condition 2 is not true since the shadow is never perfect and some field will extend behind the walls.

For $\Sigma >> \lambda$ this is an OK approximation.



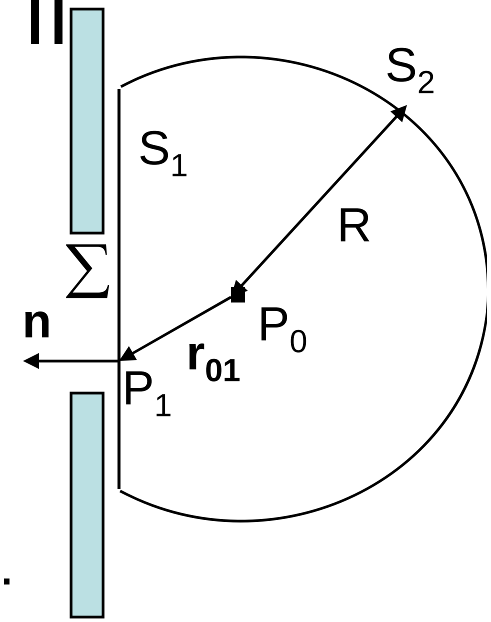
On these portions of S_1:

$$U=\frac{\partial U}{\partial n}=\frac{\partial^2 U}{\partial n^2}=0$$

Fresnel-Kirchhoff diffraction formula II

Withe above assumptions and $k >> 1/r_{01}$ we arrive at

$$U(P_0) = \frac{1}{4\pi} \iint_\Sigma \left\{ \frac{e^{jkr_{01}}}{r_{01}} \left[\frac{\partial U}{\partial n} - jkU \cos(\vec{n}, \vec{r}_{01}) \right] \right\} ds$$



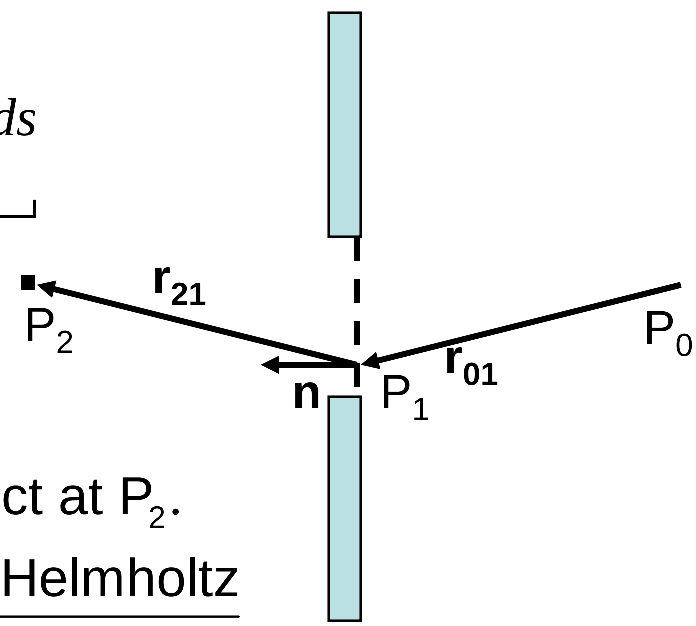
If the aperture is illuminated by a single spherical wave

$U(P_1) = \dfrac{Ae^{jkr_{21}}}{r_{21}}$ located at point P_2, at a distance r_{21} from P_1.

If $r_{21} >> \lambda$ we can show that (problem 3.3)

$$\underbrace{U(P_0) = \frac{A}{j\lambda} \iint_\Sigma \left\{ \frac{e^{jk(r_{21}+r_{01})}}{r_{21} r_{01}} \left[\frac{\cos(\vec{n}, \vec{r}_{01}) - \cos(\vec{n}, \vec{r}_{21})}{2} \right] \right\} ds}$$

The Fresnel-Kirchhoff diffraction formula that
holds only for a single point source illumination



The Fresnel-Kirchhoff diffraction formula
is symmetrical with respect to P_0 and P_2.

A point source at P_0 will produce the same effect at P_2.

This result is known as: reciprocity theorem of Helmholtz

Huygens' wavelets

Huygens-Fresnel theory: the light disturbance at a point P_0 arises from the superposition of secondary waves that produced from a surface situated between this point and the light source.

If we rewrite

$$U\left(P_0\right)=\frac{A}{j\lambda}\iint_{\Sigma}\left\{\frac{e^{jk(r_{21}+r_{01})}}{r_{21}r_{01}}\left[\frac{\cos(\vec{n},\vec{r}_{01})-\cos(\vec{n},\vec{r}_{21})}{2}\right]\right\}ds$$

$$U\left(P_0\right)=\iint_{\Sigma}U'(P_1)\frac{e^{jkr_{01}}}{r_{01}}ds$$ where

$$U'(P_1)=\frac{1}{j\lambda}\overbrace{\left[\frac{Ae^{jkr_{21}}}{r_{21}}\right]}^{\text{illuminating wavefront}}\left[\frac{\overbrace{\cos(\vec{n},\vec{r}_{01})}^{\text{Observation angle}}-\overbrace{\cos(\vec{n},\vec{r}_{21})}^{\text{Illumination angle}}}{2}\right]$$

Seems like $U(P_0)$ arises from sum of infinite fictitious sources with amplitudes and phases expressed by $U'(P_1)$.

We used point source to get this result but it is possible to generalize this result for any illlumination by using Rayleigh-Sommerfeld theory.

The Rayleigh-Sommerfeld Formulation of diffraction

Potential theory: If a two-dimensional potential function and its normal derivaive vanish together along any finite curve segment, then the potential function must vanish over the entire plane.

This is also true for solution of a three-dimensional wave equation.

1) The Kirchhoff boundary conitions suggests that the diffracted field must be zero everywhere behind the aperture. Not true.

2) Also close to aperture the theory fails to produce the observed diffraction field.

Inconsistencies of the Kirchhoff theory were removed by Sommerfeld. He elliminated the need of imposing boundary values on the disturbance and its derivative simultaneously.

An alternative Green's function I

Observed field strength in terms of the incident field and its normal derivatives:

$$U(P_0) = \frac{1}{2\pi}\iint_{S_1}\left(\frac{\partial U}{\partial n}G - U\frac{\partial G}{\partial n}\right)ds \qquad \text{(Fresnel-Kirchhoff diffraction formula)}$$

Conditions for validity:

1) The scalar theory holds

2) Both U and G are solutions of the homogeneous scalar wave equation

3) The sommerfeld radiation condition holds i.e. if the disturbance U vanishes at least as fast as a diverging spherical wave then: as $R \to \infty$ contribution of S_2 to $U(P_0)$ vanishes.

If the Green's function of Kirchhoff theory was modified so that either G or $\partial G / \partial n$ vanished over entire surface S_1, then there is no need to impose boundary condtions on both U and $\partial U / \partial n$.

An alternative Green's function II

Somerfeld argued that one Green's function that meets these criteria is composed of two identical point sourcers at two sides of the aperture, mirror image of each other, oscillating with a 180^0 phase difference:

$$G_{-}(P_1) = \frac{e^{jkr_{01}}}{r_{01}} - \frac{e^{jk\tilde{r}_{01}}}{\tilde{r}_{01}}$$

Now $G = 0$ on the plane of aperture.

Kirchhoff's BC may be applied only on U

$$U(P_0) = \frac{1}{2\pi}\iint_{S_1}\left(\frac{\partial U}{\partial n}G - U\frac{\partial G}{\partial n}\right)ds$$

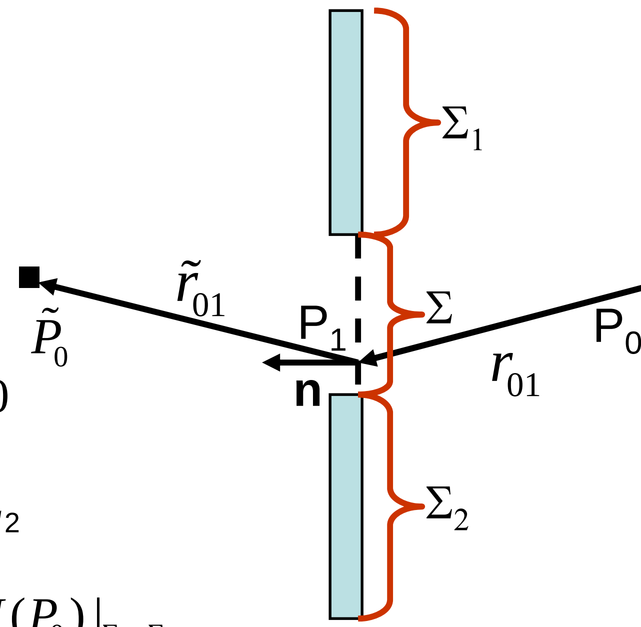
$$S_1 = \Sigma_1 + \Sigma + \Sigma_2$$

we need $U(P_0) = \frac{1}{2\pi}\iint_{\Sigma_1+\Sigma_2}\left(\frac{\partial U}{\partial n}G - U\frac{\partial G}{\partial n}\right)ds = 0$

$G|_{S_1} = 0$ so if we require only $U = 0$ on Σ_1 and Σ_2

then $\frac{\partial U}{\partial n}$ does not need to be zero to make $U(P_0)|_{\Sigma_1+\Sigma_2}$

With the new BC on the U only there is no conflict with the potential theorem.



The Rayleigh-Sommerfeld diffraction Formula

With the Green's function G_-

$$G_-(P_1) = \frac{e^{jkr_{01}}}{r_{01}} - \frac{e^{jk\tilde{r}_{01}}}{\tilde{r}_{01}}$$

the $U\left(P_0\right)$ takes the form

$$U_I\left(P_0\right) = \frac{1}{j\lambda}\iint_{S_1} U(P_1)\frac{e^{jkr_{01}}}{r_{01}}\cos(\vec{n},\vec{r}_{01})ds \text{ or}$$

Assuming $r_{01} >> \lambda$

Now applying the Kirchhoff BC only on U and not on $\partial U / \partial n$ we get

$$U_I\left(P_0\right) = \frac{1}{j\lambda}\iint_{\Sigma} U(P_1)\frac{e^{jkr_{01}}}{r_{01}}\cos(\vec{n},\vec{r}_{01})ds$$

And U will not vanish on the other side of the aperture.

Rayleigh-Sommerfeld Diffraction formula

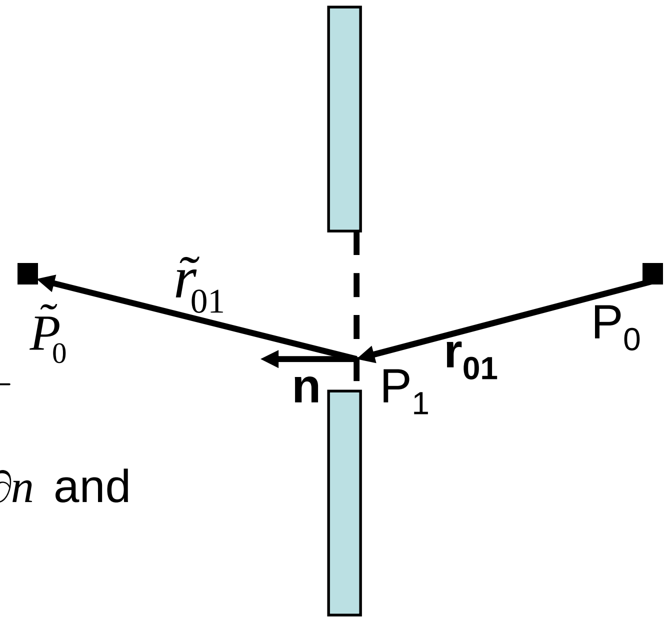
With the Green's function G_+ two sources oscillating in phase with each other

$$G_+(P_1) = \frac{e^{jkr_{01}}}{r_{01}} + \frac{e^{jk\tilde{r}_{01}}}{\tilde{r}_{01}}$$ the $U\left(P_0\right)$ takes the form

$$U_{II}\left(P_0\right) = \frac{1}{2\pi}\iint_\Sigma \frac{\partial U(P_1)}{\partial n}\frac{e^{jkr_{01}}}{r_{01}}ds$$

Now for the spacial case illumination:

a diverging spherical wave from point P_2: $U(P_1) = A\frac{e^{jkr_{21}}}{r_{21}}$



We apply the Kirchhoff BC only on U and not on $\partial U / \partial n$ and using G_- we get

$$U_I\left(P_0\right) = \frac{A}{j\lambda}\iint_\Sigma \frac{e^{jk(r_{21}+r_{01})}}{r_{21}r_{01}}\cos(\vec{n},\vec{r}_{01})ds$$

and and G_+ gives

$$U_{II}\left(P_0\right) = \frac{A}{j\lambda}\iint_\Sigma \frac{e^{jk(r_{21}+r_{01})}}{r_{21}r_{01}}\cos(\vec{n},\vec{r}_{21})ds$$

Where the angle between $\vec{n}$ and $\vec{r}_{21}$ is greater than 90^0.

This is Rayleigh-Sommerfeld Diffraction formula where we assumed $r_{21} >> \lambda$

Comparison of the Kirchhoff and Rayleigh-Sommerfeld (R-S) theorem

$$\underbrace{G_-(P_1)=\frac{e^{jkr_{01}}}{r_{01}}-\frac{e^{jk\tilde{r}_{01}}}{\tilde{r}_{01}};\;\; G_+(P_1)=\frac{e^{jkr_{01}}}{r_{01}}+\frac{e^{jk\tilde{r}_{01}}}{\tilde{r}_{01}},}_{\text{Green functions of the Sommerfeld formulation}} \qquad \underbrace{G_K(P_1)=\frac{e^{jkr_{01}}}{r_{01}}}_{\text{Green function of the Kirchhoff formulation}}$$

On the surface Σ we can show that $G_+ = 2G_K$ and $\frac{\partial G_-}{\partial n}=2G_K$

For the Kirchhoff theory: $$U(P_0)=\frac{1}{4\pi}\iint_\Sigma\left(\frac{\partial U}{\partial n}G_K-U\frac{\partial G_K}{\partial n}\right)ds$$

For the R-S theory: $$U_I(P_0)=\frac{-1}{2\pi}\iint_\Sigma U\frac{\partial G_K}{\partial n}ds$$

$$U_{II}(P_0)=\frac{1}{2\pi}\iint_\Sigma \frac{\partial U}{\partial n}G_K ds$$

We can see that

$$\boxed{U(P_0)=\frac{U_I(P_0)+U_{II}(P_0)}{2}}$$

Summary: the Kirchhoff solution is the arithmatic average of the two Rayleigh-Sommerfeld solutions.

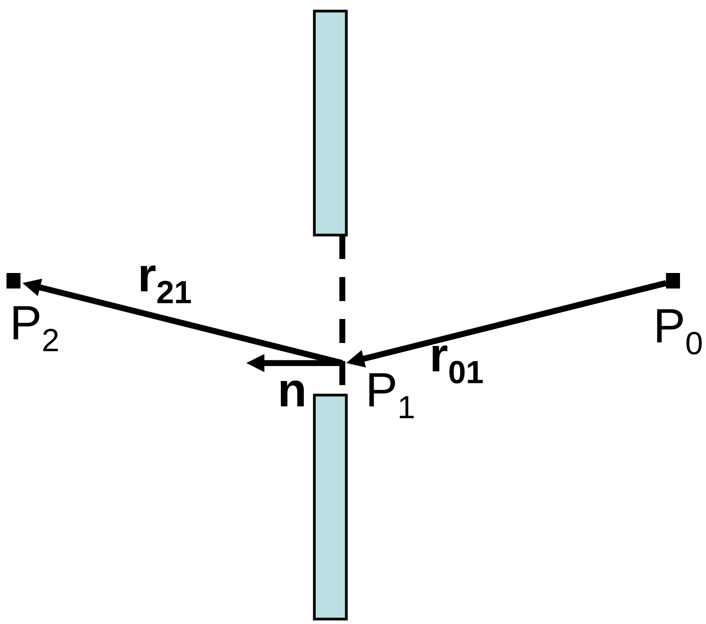
Comparison of the Kirchhoff and R-S theory

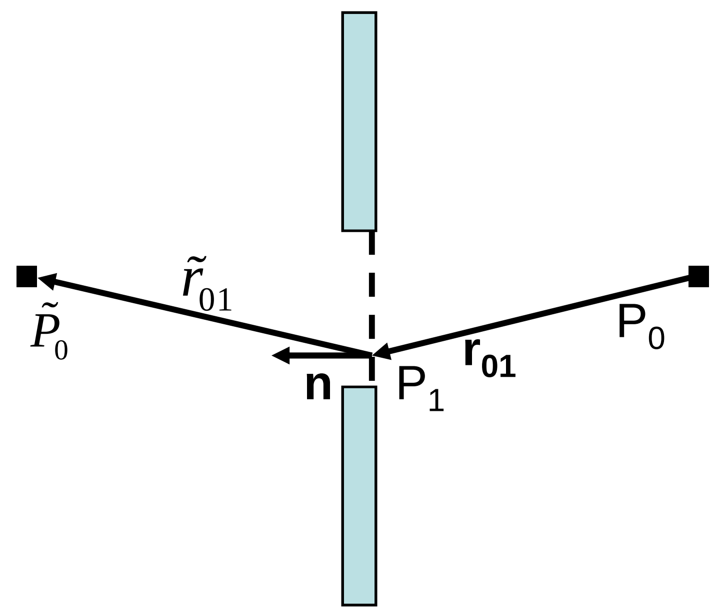
Kirchhoff theory:

$$U\left(P_0\right)=\frac{A}{j\lambda}\iint_{\Sigma}\frac{e^{jk(r_{21}+r_{01})}}{r_{21}r_{01}}\overbrace{\left[\frac{\cos(\vec{n},\vec{r}_{01})-\cos(\vec{n},\vec{r}_{21})}{2}\right]}^{\text{Obliquity factor } \psi}ds=\frac{A}{j\lambda}\iint_{\Sigma}\frac{e^{jk(r_{21}+r_{01})}}{r_{21}r_{01}}\psi ds$$

R-S theory: $U_I\left(P_0\right)=\frac{A}{j\lambda}\iint_{\Sigma}\frac{e^{jk(r_{21}+r_{01})}}{r_{21}r_{01}}\overbrace{\cos(\vec{n},\vec{r}_{01})}^{\text{Obliquity factor } \psi}\ ds=\frac{A}{j\lambda}\iint_{\Sigma}\frac{e^{jk(r_{21}+r_{01})}}{r_{21}r_{01}}\psi ds$

$$U_{II}\left(P_0\right)=\frac{A}{j\lambda}\iint_{\Sigma}\frac{e^{jk(r_{21}+r_{01})}}{r_{21}r_{01}}\overbrace{\cos(\vec{n},\vec{r}_{21})}^{\text{Obliquity factor } \psi}\ ds=\frac{A}{j\lambda}\iint_{\Sigma}\frac{e^{jk(r_{21}+r_{01})}}{r_{21}r_{01}}\psi ds$$

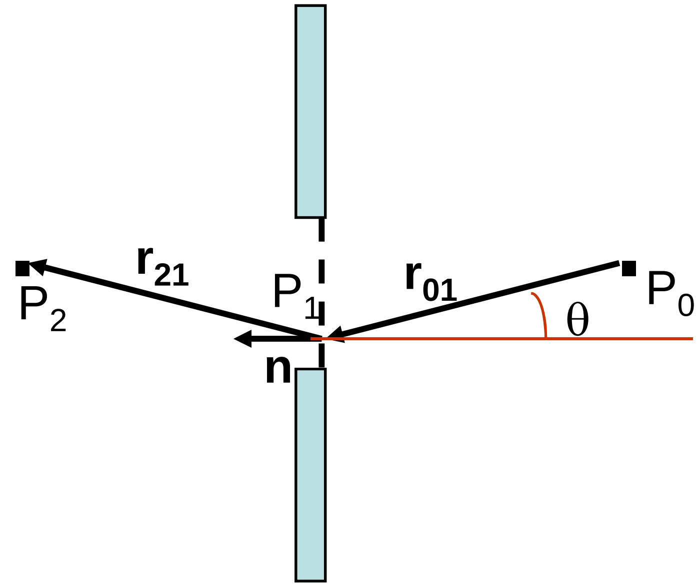




Comparison of the Kirchhoff and R-S theory

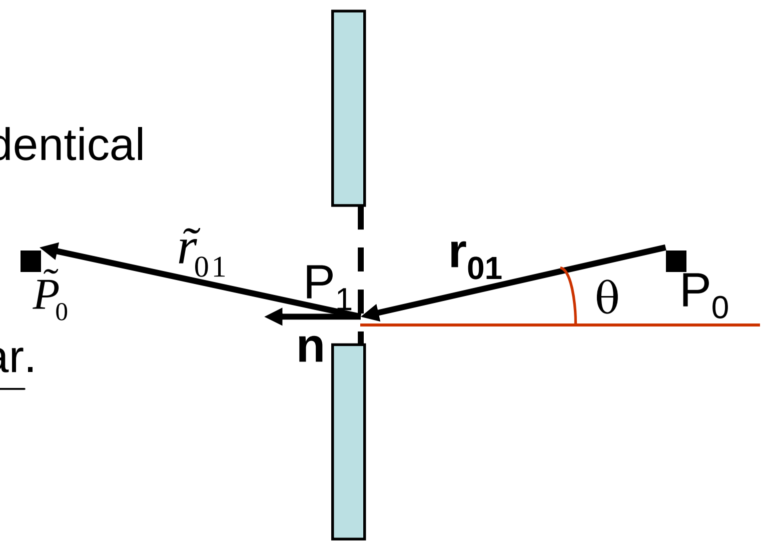
Obliquity factor of both Kirchhoff and R-S theory

$$\psi=\begin{cases} \frac{1}{2}[\cos(\vec{n},\vec{r}_{01})-\cos(\vec{n},\vec{r}_{21})] & \text{Kirchhof theory} \\ \cos(\vec{n},\vec{r}_{01}) & \text{First R-S solution} \\ -\cos(\vec{n},\vec{r}_{21}) & \text{Second R-S solution} \end{cases}$$



When a point source is at a very far distance

$$\psi=\begin{cases} \frac{1}{2}[1+\cos\theta] & \text{Kirchhof theory} \\ \cos(\theta) & \text{First R-S solution} \\ 1 & \text{Second R-S solution} \end{cases}$$



In summary: for small angles all three solutions are identical

When observation point or illumination source are far away, the angles are small.

R-S solution requires the diffracting screens be planar.

Kichhoff solution is not limitted to planar surfaces.

For most applications both are OK.

We will use the first R-S solution for simplicity.

Huygens-Fresnel Principle

Generalization to Non-monochromatic waves I

We generalize the R-S's first solution to nonmonochromatic waves (chromatic?)

Monochromatic time function of the disturbance:

$$u(P,t) = \mathrm{Re}\left\{U(P)e^{-j2\pi\nu t}\right\}$$

Time dependent chromatic functions: $u(P_1,t)$ at the aperture, $u(P_0,t)$ observation point

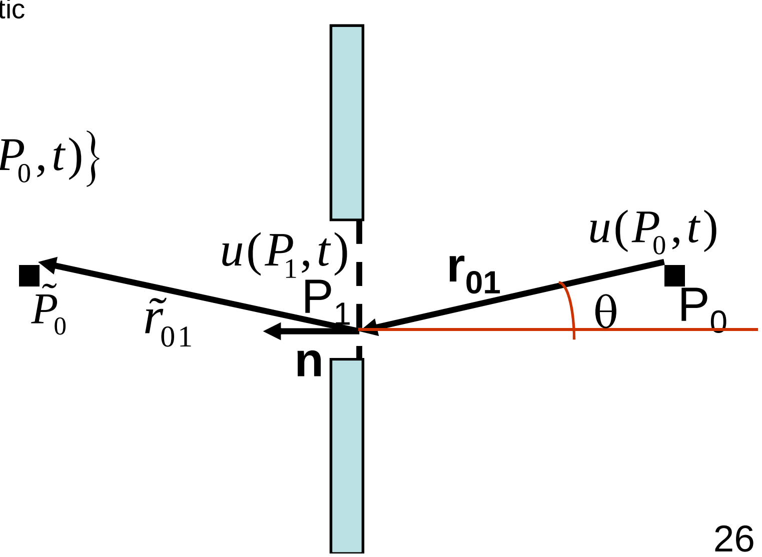
$u(P_1,t),\ u(P_0,t)$ in terms of their Fourier transforms. Let's change the variable $\nu' = -\nu$

$$u(P_1,t) = \int_{-\infty}^{\infty} U(P_1,\nu)e^{j2\pi\nu t}d\nu = \int_{-\infty}^{\infty} U(P_1,-\nu')e^{-j2\pi\nu' t}d\nu'$$

$$u(P_0,t) = \int_{-\infty}^{\infty} U(P_0,\nu)e^{j2\pi\nu t}d\nu = \int_{-\infty}^{\infty} \underbrace{U(P_0,-\nu')}_{\text{complex amplitudes of the disturbance at frequency } \nu'} \underbrace{e^{-j2\pi\nu' t}}_{\text{monochromatic elementary function of freuency } \nu'} d\nu'$$

Where $U(P_1,\nu) = \mathcal{F}\left\{u(P_1,t)\right\}$ and $U(P_0,\nu) = \mathcal{F}\left\{u(P_0,t)\right\}$

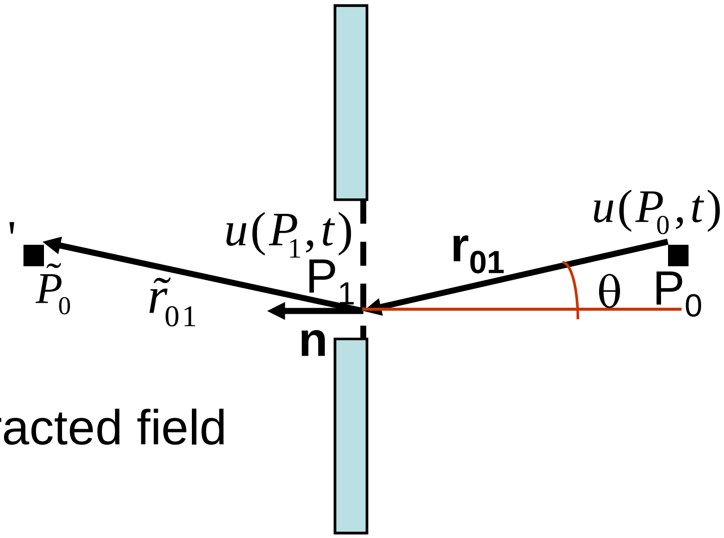
We see that the cromatic function is sum of the monochromatic functions over different frequencies.



Generalization to Non-monochromatic waves II

$$u(P_1,t)=\int_{-\infty}^{\infty}U(P_1,\nu)e^{j2\pi\nu t}d\nu=\int_{-\infty}^{\infty}U(P_1,-\nu')e^{-j2\pi\nu' t}d\nu'$$

$$u(P_0,t)=\int_{-\infty}^{\infty}U(P_0,\nu)e^{j2\pi\nu t}d\nu=\int_{-\infty}^{\infty}\underbrace{U(P_0,-\nu')}_{\text{complex amplitude of the disturbance at frequency } \nu'}\ \underbrace{e^{-j2\pi\nu' t}}_{\text{elementary function of freuency } \nu'}\ d\nu'$$



Now we introduce the R-S first solution for the diffracted field

$$U_I\left(P_0\right)=\frac{1}{j\lambda}\iint_\Sigma \frac{\overbrace{Ae^{jkr_{21}}}^{U(P_1)}}{r_{21}}\frac{e^{jkr_{01}}}{r_{01}}\overbrace{\cos(\vec{n},\vec{r}_{01})}^{\text{Obliquity factor } \psi}\ ds=\frac{1}{j\lambda}\iint_\Sigma U(P_1)\frac{e^{jkr_{01}}}{r_{01}}\cos(\vec{n},\vec{r}_{01})ds$$

$$U\left(P_0,-\nu'\right)=-j\frac{\nu'}{V}\iint_\Sigma U(P_1,-\nu')\frac{e^{j2\pi\nu' r_{01}/V}}{r_{01}}\cos(\vec{n},\vec{r}_{01})ds \text{ where } V=c/n$$

This is <u>one frequency component</u>. Summing over all of them we get:

$$u\left(P_0,t\right)=\int_{-\infty}^{\infty}\overbrace{\left[-j\frac{\nu'}{V}\iint_\Sigma U(P_1,-\nu')\frac{e^{j2\pi\nu' r_{01}/V}}{r_{01}}\cos(\vec{n},\vec{r}_{01})ds\right]}^{\text{Complex amplitude at each frequency}}\ \overbrace{e^{-j2\pi\nu' t}}^{\text{Elementary function at that frequency}}\ d\nu'$$

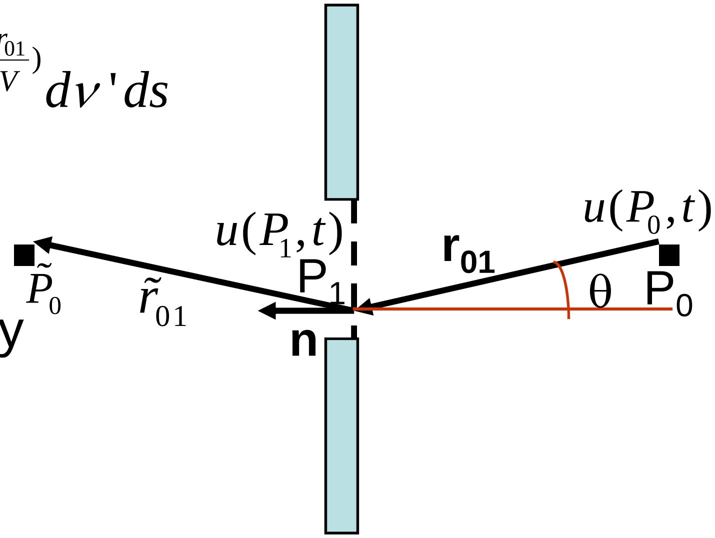
$$u\left(P_0,t\right)=\iint_\Sigma\frac{\cos(\vec{n},\vec{r}_{01})}{2\pi V r_{01}}\int_{-\infty}^{\infty}-j2\pi\nu' U(P_1,-\nu')e^{-j2\pi\nu'(t-\frac{r_{01}}{V})}d\nu' ds$$

Generalization to Non-monochromatic waves III

Next we want to relate the disturbance at the observation point

$$u(P_0,t) = \iint_\Sigma \frac{\cos(\vec{n},\vec{r}_{01})}{2\pi V r_{01}} \int_{-\infty}^{\infty} -j2\pi\nu' U(P_1,-\nu') e^{-j2\pi\nu'(t-\frac{r_{01}}{V})} d\nu' ds$$

to the disturbance at the aperture location



$u(P_1,t) = \int_{-\infty}^{\infty} U(P_1,-\nu')e^{-j2\pi\nu' t} d\nu'$ we use the identity

$$\frac{d}{dt}u(P_1,t) = \frac{d}{dt}\int_{-\infty}^{\infty} U(P_1,-\nu')e^{-j2\pi\nu' t} d\nu'$$

$$\frac{d}{dt}u(P_1,t) = \int_{-\infty}^{\infty} -j2\pi\nu' U(P_1,-\nu')e^{-j2\pi\nu' t} d\nu'$$

$$\underbrace{u(P_0,t)}_{\substack{\text{The wave disturbance at } P_0 \text{ is linearly proportional}\\ \text{to the time derivative of the disturbance at eah point } P_1 \\ \text{on the aperture}}} = \iint_\Sigma \underbrace{\frac{\cos(\vec{n},\vec{r}_{01})}{2\pi V r_{01}}}_{\text{Over all angles}} \frac{d}{dt} \underbrace{u(P_1, t - \frac{r_{01}}{V})}_{\substack{\text{Incident wave at the "retarded" time}\\ \text{or the time that the wave was generated}}} ds$$

<u>In summary</u> the results of the diffration theory for the monochromatic waves is applicable to the more general case of the chromatic waves.

The angular spectrum of plane a wave I

Next we want to formulate the diffraction theory in a framework of linear, invariant systems.

Assume a transverse monochomatic wave traveleing in $+z$ direction incident on a transverse $(x,\ y)$ plane

Across the $z=0$ plane $U=U(x,y,0)$

Across the $z=z$ plane $U=U(x,y,z)$

Objective: to calculate the resulting field $U=U(x,y,z)$ down the road $z>0$ as a function of $U(x,y,0)$

FT of the U at $z=0$ plane: $A(f_X,f_Y;0)=\iint_{-\infty}^{\infty}U(x,y,0)e^{-j2\pi(f_X x+f_Y y)}dxdy$

And $U(x,y,0)=\iint_{-\infty}^{\infty}\underbrace{A(f_X,f_Y;0)}_{\text{What is the physical meaning of these components?}}e^{j2\pi(f_X x+f_Y y)}df_X df_Y$

So far we have looked at $A(f_X,f_Y;0)$ as the spatial frequency spectrum of the disturbance.

What is the direction of propagation of each these components?

Physical interpretation of angular spectrum

Consider a simple plane wave propagating in direction of k:

$P(x,y,z)=e^{j\mathbf{k}.\mathbf{r}}$ where $\mathbf{r}=x\hat{x}+y\hat{y}+z\hat{z}$ and $\mathbf{k}=\dfrac{2\pi}{\lambda}(\alpha\hat{x}+\beta\hat{y}+\gamma\hat{z})$;

$\alpha,\ \beta$, and γ are the direction cosines of $\mathbf{k}$. Also $|\mathbf{k}|=\dfrac{2\pi}{\lambda}$

Using $\alpha^2+\beta^2+\gamma^2=1$, between the direction cosines we rewrite

$$\alpha=\frac{k_x}{|\mathbf{k}|}=\frac{2\pi/\lambda_x}{2\pi/\lambda}=\frac{\lambda}{\lambda_x}=f_X\lambda;\ \beta=f_Y\lambda;\ \gamma=f_Z\lambda=\sqrt{1-\left(f_X\lambda\right)^2-\left(f_Y\lambda\right)^2}$$

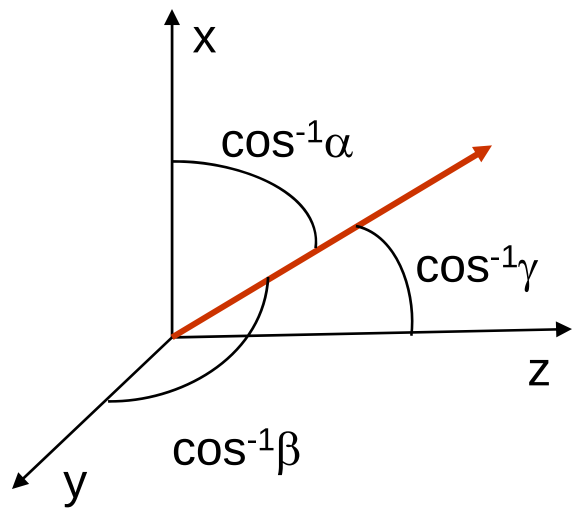
$$P(x,y,z)=e^{jk.r}=e^{j\frac{2\pi}{\lambda}(\alpha x+\beta y)}e^{j\frac{2\pi}{\lambda}\gamma z}=e^{j2\pi(f_X x+f_y y)}e^{j2\pi\sqrt{1-f_X^2-f_Y^2}}$$

Now with $f_X=\alpha/\lambda,\ f_Y=\beta/\lambda\ f_z=0$ we can write

$A(f_X,f_Y;0)=\iint_{-\infty}^{\infty}U(x,y,0)e^{-j2\pi(f_X x+f_Y y)}dxdy$ as

$$A\left(\frac{\alpha}{\lambda},\frac{\beta}{\lambda};0\right)=\iint_{-\infty}^{\infty}U(x,y,0)e^{-j2\pi\left(\frac{\alpha}{\lambda}x+\frac{\beta}{\lambda}y\right)}dxdy$$

is the angular spectrum of the disturbance $U(x,y,0)$.



In summary this results shows that:

each spatial frequency component is propagating at a different angle.

Propagation of the angular spectrum I

Consider the angular spectrum of the U across a plane parallel to (x, y) at a distance z from it:

$$A\left(\frac{\alpha}{\lambda},\frac{\beta}{\lambda};z\right)=\iint_{-\infty}^{\infty}U(x,y,z)e^{-j2\pi\left(\frac{\alpha}{\lambda}x+\frac{\beta}{\lambda}y\right)}dxdy$$

Our goal is to find the effects of the wave propagation on the angular spectrum of the disurbance or the relationship of $A\left(\frac{\alpha}{\lambda},\frac{\beta}{\lambda};0\right)$ and $A\left(\frac{\alpha}{\lambda},\frac{\beta}{\lambda};z\right)$

We start from $U(x,y,z)=\iint_{-\infty}^{\infty}A\left(\frac{\alpha}{\lambda},\frac{\beta}{\lambda};z\right)e^{j2\pi\left(\frac{\alpha}{\lambda}x+\frac{\beta}{\lambda}y\right)}d\frac{\alpha}{\lambda}d\frac{\beta}{\lambda}$

U must satisfy the Helmholtz equation $\nabla^2 U+k^2U=0$ wherever there is no source.

The result is that A must satisfy the following differential equation.

$$\frac{d^2}{dz^2}A\left(\frac{\alpha}{\lambda},\frac{\beta}{\lambda};z\right)+\underbrace{\left(\frac{2\pi}{\lambda}\right)^2\left[1-\alpha^2-\beta^2\right]}_{\text{coefficient}}A\left(\frac{\alpha}{\lambda},\frac{\beta}{\lambda};z\right)=0$$

The solution can have the form:

$$A\left(\frac{\alpha}{\lambda},\frac{\beta}{\lambda};z\right)=A\left(\frac{\alpha}{\lambda},\frac{\beta}{\lambda};0\right)e^{\left(j\frac{2\pi}{\lambda}\sqrt{1-\alpha^2-\beta^2}\,z\right)}$$

Propagation of the angular spectrum II

$$A\left(\frac{\alpha}{\lambda},\frac{\beta}{\lambda};Z\right)=A\left(\frac{\alpha}{\lambda},\frac{\beta}{\lambda};0\right)e^{\left(j\frac{2\pi}{\lambda}\sqrt{1-\alpha^2-\beta^2}\,z\right)}$$

For this solution two cases are recognized:

1) When $\alpha^2+\beta^2<1$ (true for all direction cosines) the effect of propagation on the angular spectrum is simply a change in phase of each component.

2) When $\alpha^2+\beta^2>1$ (α and β are no longer the direction cosines effect of aperture is present here) the angular spectrum has the form:

$$A\left(\frac{\alpha}{\lambda},\frac{\beta}{\lambda};z\right)=\overbrace{A\left(\frac{\alpha}{\lambda},\frac{\beta}{\lambda};0\right)}^{\text{Fourier transform of a field distributiion on which BCs of the aperture is imposed}}e^{\left(\overbrace{j\frac{2\pi}{\lambda}\sqrt{1-\alpha^2-\beta^2}}^{\text{A real number}}\,z\right)}=A\left(\frac{\alpha}{\lambda},\frac{\beta}{\lambda};0\right)e^{-\mu z};\ \mu=\frac{2\pi}{\lambda}\sqrt{\alpha^2+\beta^2-1}$$

Since z is a positive real number, the wave components are attenuating as they propagate. They are also called evanescent waves.

Simmilar to the case of microwave waveguides there is a cutoff frequency. Below cutoff frequency, these evanescent waves carry no energy away from the aperture.

Propagation of the angular spectrum III

Substituting $A\left(\frac{\alpha}{\lambda},\frac{\beta}{\lambda};Z\right)=A\left(\frac{\alpha}{\lambda},\frac{\beta}{\lambda};0\right)e^{\left(j\frac{2\pi}{\lambda}\sqrt{1-\alpha^2-\beta^2}z\right)}$ in the $U(x,y,z)$ we get:

$$U(x,y,z)=\iint_{-\infty}^{\infty}A\left(\frac{\alpha}{\lambda},\frac{\beta}{\lambda};Z\right)e^{j2\pi\left(\frac{\alpha}{\lambda}x+\frac{\beta}{\lambda}y\right)}d\frac{\alpha}{\lambda}d\frac{\beta}{\lambda}$$

$$U(x,y,z)=\iint_{-\infty}^{\infty}A\left(\frac{\alpha}{\lambda},\frac{\beta}{\lambda};0\right)e^{\left(j\frac{2\pi}{\lambda}\sqrt{1-\alpha^2-\beta^2}z\right)}e^{j2\pi\left(\frac{\alpha}{\lambda}x+\frac{\beta}{\lambda}y\right)}d\frac{\alpha}{\lambda}d\frac{\beta}{\lambda}$$

$$circ\left(\sqrt{x^2+y^2}\right)=\begin{cases}1 & \sqrt{x^2+y^2}<1\\ 1/2 & \sqrt{x^2+y^2}=1\\ 0 & otherwise\end{cases}$$

$$U(x,y,z)=\iint_{-\infty}^{\infty}A\left(\frac{\alpha}{\lambda},\frac{\beta}{\lambda};0\right)e^{\left(j\frac{2\pi}{\lambda}\sqrt{1-\alpha^2-\beta^2}z\right)}circ\sqrt{\alpha^2+\beta^2}\,e^{j2\pi\left(\frac{\alpha}{\lambda}x+\frac{\beta}{\lambda}y\right)}d\frac{\alpha}{\lambda}d\frac{\beta}{\lambda}$$

No angular spectrum contribute to $U(x,y,z)$ <u>beyond</u> the evanescent wave <u>cutoff</u> $\sqrt{\alpha^2+\beta^2}<1$.

Physical meaning of cutoff frequency

No angular spectrum contribute to $U(x, y, z)$ beyond the evanescent wave cutoff $\sqrt{\alpha^2 + \beta^2} < 1$.

No imaging system can resolve a periodic structure that its period is less than the wavelength of the light used.

$$\sqrt{\left(\frac{k_x}{k}\right)^2 + \left(\frac{k_y}{k}\right)^2} < 1 \rightarrow \sqrt{k_x^2 + k_y^2} < |k| \quad \text{or} \quad \sqrt{x^2 + y^2} > |\lambda|.$$

Near-Field imaging couples to the evanescent waves of a very fine structure and recovers the phase information that would be lost otherwise.

Effects of a diffracting aperture on the angular spectrum I

Now an infinite opaque screen containing a diffracting structure is placesd in the $z = 0$ plane.

Goal: find effects of the screen on the angular spectrum of the disturbance.

Amplitude transmitance function:

$$t_A(x,y) = \frac{U_t(x,y;0)}{U_i(x,y;0)}$$

$U_t(x,y;0) = t_A(x,y)U_i(x,y;0)$ take the Fourier transform of the both sides and use the frequency convolution theorem:

$$\underbrace{A_t\left(\frac{\alpha}{\lambda},\frac{\beta}{\lambda}\right)}_{\substack{\text{Angular spectrum}\\\text{of the transmitted}\\\text{disturbance}}} = \underbrace{A_i\left(\frac{\alpha}{\lambda},\frac{\beta}{\lambda}\right)}_{\substack{\text{Angular spectrum}\\\text{of the incident}\\\text{disturbance}}} \underbrace{\otimes}_{\substack{\text{Convolved}\\\text{with}}} \underbrace{T\left(\frac{\alpha}{\lambda},\frac{\beta}{\lambda}\right)}_{\substack{\text{A second angular}\\\text{spectrum that is a}\\\text{result of the diffracting}\\\text{structure}}}$$

where $T\left(\frac{\alpha}{\lambda},\frac{\beta}{\lambda}\right) = \int\int_{-\infty}^{\infty} t_A(x,y)e^{-j2\pi\left(\frac{\alpha}{\lambda}x+\frac{\beta}{\lambda}y\right)}dxdy$

is the Fourier transform of the $t_A(x,y)$

Effects of a diffracting aperture on the angular spectrum II

Example: for a unit amplitude plane wave illuminating the diffracting structure, the angular spectrum of the input is a delta function:

$$A_i\left(\frac{\alpha}{\lambda},\frac{\beta}{\lambda}\right)=\delta\left(\frac{\alpha}{\lambda},\frac{\beta}{\lambda}\right) \qquad A_t\left(\frac{\alpha}{\lambda},\frac{\beta}{\lambda}\right)=A_i\left(\frac{\alpha}{\lambda},\frac{\beta}{\lambda}\right)\otimes T\left(\frac{\alpha}{\lambda},\frac{\beta}{\lambda}\right)$$

$$\overbrace{A_t\left(\frac{\alpha}{\lambda},\frac{\beta}{\lambda}\right)}^{\text{Transmitted angular spectrum}}=\delta\left(\frac{\alpha}{\lambda},\frac{\beta}{\lambda}\right)\otimes T\left(\frac{\alpha}{\lambda},\frac{\beta}{\lambda}\right)=\overbrace{T\left(\frac{\alpha}{\lambda},\frac{\beta}{\lambda}\right)}^{\text{Fourier transform of the amplitude transmitance function of the aperture}}$$

If the diffracting structure is an aperture that limits the extend of the field distribution,

$$T\left(\frac{\alpha}{\lambda},\frac{\beta}{\lambda}\right)=\int\int_{-\infty}^{\infty}t_A(x,y)e^{-j2\pi\left(\frac{\alpha}{\lambda}x+\frac{\beta}{\lambda}y\right)}dxdy$$

the angular spectrum of the disturbance will be broadened.

The propagation phenomenon as a linear filter I

Consider propagation from plane $z = 0$ to the plane $z = z$

Across the $z = 0$ plane $U = U(x, y, 0)$

Across the $z = z$ plane $U = U(x, y, z)$

Goals: a) to show that the propagation phenomenon acts like a linear space-invariant suystem.

b) find the system's transfer function

The system is linear since it is governed by a linear wave equation or considering the superposition integral (R-S first solution).

$$U_I(P_0) = \frac{1}{j\lambda}\iint_{\Sigma} U(P_1)\frac{e^{jkr_{01}}}{r_{01}}\cos(\vec{n}, \vec{r}_{01})ds = \frac{1}{j\lambda}\iint_{\Sigma} U(P_1)\frac{e^{jkr_{01}}}{r_{01}}\cos\theta ds$$

$$U(P_0) = \iint_{\Sigma} h(P_0, P_1)U(P_1)ds \quad \text{with} \quad h(P_0, P_1) = \frac{1}{j\lambda}\frac{e^{jkr_{01}}}{r_{01}}\cos\theta$$

To establish the space-invariance we need to derive the transfer function of the system and show that the mapping is space-invariant.

Transfer function of the linear invariant systems

$g_2(x_2, y_2) = S\left\{g_1(x_1, y_1\right\}$ action of a linear operator on a input

$g_2(x_2, y_2) = \int\int_{-\infty}^{\infty} g_1(\xi, \eta) h(x_2, y_2; \xi, \eta) d\xi d\eta$ h is the impulse response

$$h(x_2, y_2; \xi, \eta) = S\{\delta(x_1 - \xi; y_1 - \eta)\}$$

$h(x_2, y_2; \xi, \eta) = h(x_2 - \xi,\ y_2 - \eta)$ for linear invariant systems

$$g_2(x_2, y_2) = \int\int_{-\infty}^{\infty} \underbrace{g_1(\xi, \eta)}_{\text{Object function}} \ \underbrace{h(x_2 - \xi, y_2 - \eta)}_{\text{Impulse response of the system}} d\xi d\eta = g_1 * h = g_1 \otimes h$$

Take Fourier transform

$$G_2(f_X, f_Y) = H(f_X, f_Y) G_1(f_X, f_Y)$$

Where H is the Fourier transform of the impulse response.

$$H(f_X, f_Y) = \int\int_{-\infty}^{\infty} h(\xi, \eta) e^{-j2\pi(f_X \xi + f_Y \eta)} d\xi d\eta$$

And it is called <u>transfer function</u> of the system that indicates effects of the system in the freuency domain.

Propagation phenomenon as a linear filter II

To establish the space-invariance we need to derive the transfer function of the system and show that the mapping is space-invariant.

Let $A(f_X, f_Y; 0)$ be spatial spectrum (Fourier transform) of $U(x, y, 0)$ at $z = 0$

Let $A(f_X, f_Y; z)$ be spatial spectrum (Fourier transform) of $U(x, y, z)$ at $z = z$

So following $G_2(f_X, f_Y) = H(f_X, f_Y) G_1(f_X, f_Y)$ and $A(f_X, f_Y; z) = H(f_X, f_Y) A(f_X, f_Y; 0)$ we can see H connects the two frequency spectrums before propagation and after the propagation. We got

$$U(x,y,z) = \int\int_{-\infty}^{\infty} A\left(\frac{\alpha}{\lambda}, \frac{\beta}{\lambda}; 0\right) e^{\left(j\frac{2\pi}{\lambda}\sqrt{1-\alpha^2-\beta^2}\, z\right)} circ\sqrt{\alpha^2+\beta^2}\, e^{j2\pi\left(\frac{\alpha}{\lambda}x + \frac{\beta}{\lambda}y\right)} d\frac{\alpha}{\lambda}\, d\frac{\beta}{\lambda}$$

with $\frac{\alpha}{\lambda} = f_X, \frac{\beta}{\lambda} = f_Y$

$$U(x,y,z) = \int\int_{-\infty}^{\infty} A(f_X, f_Y; 0)\, \underbrace{circ\sqrt{(\lambda f_X)^2 + (\lambda f_Y)^2}}_{\text{Imposes the bandwidth limitation associated with evanescent waves}}\, e^{\left(j\frac{2\pi}{\lambda}\sqrt{1-(\lambda f_X)^2-(\lambda f_Y)^2}\, z\right)} e^{j2\pi(f_X x + f_Y y)} df_X\, df_Y$$

$$U(x,y,z) = \int\int_{-\infty}^{\infty} A(f_X, f_Y; z) e^{j2\pi(f_X x + f_Y y)} df_X\, df_Y$$ comparing two equations shows:

$$A(f_X, f_Y; z) = A(f_X, f_Y; 0)\, circ\sqrt{(\lambda f_X)^2 + (\lambda f_Y)^2}\, e^{\left(j\frac{2\pi}{\lambda}\sqrt{1-(\lambda f_X)^2-(\lambda f_Y)^2}\, z\right)}$$

Propagation phenomenon as linear filter III

$$A(f_X, f_Y; z) = A\left(f_X, f_Y; 0\right) circ\sqrt{\left(\lambda f_X\right)^2 + (\lambda f_Y)^2}\, e^{\left(j\frac{2\pi z}{\lambda}\sqrt{1-(\lambda f_X)^2-(\lambda f_Y)^2}\right)}$$

And the transfer function is then

$$H(f_X, f_Y) = \begin{cases} e^{\left(j\frac{2\pi z}{\lambda}\sqrt{1-(\lambda f_X)^2-(\lambda f_Y)^2}\right)} & \sqrt{\left(\lambda f_X\right)^2 + (\lambda f_Y)^2} < 1/\lambda \\ 0 & \text{otherwise} \end{cases}$$

This shows that the propagation phenonmenon can be considered as a linear, dispersive spatial filter with a finite bandwidth.

Transmission is zero outside the circular region of radius λ^{-1}(in spatial frequency space)

Its transfer function is exponential.

Within the circular frequency bandwidth the modulus (or amplitude) of the function is 1, but frequency dependent phase shifts are introduced.

Phase dispersion is largest at high spatial frequencies (below the cutoff)

As $f_X \to 0$ and $f_Y \to 0$ then phase dispersion vanishes.

For a fixed spatial frequency pair, f_X and f_Y:

the phase dispersion increases as z increases