Wave Optics Analysis of Coherent Optical Systems

Appendix B Introduction to Paraxial Geometrical Optics

B.1 The Domain of Geometrical Optics

B.2 Refraction, Snell's Law, and the Paraxial Approximation

B.3 The Ray Transfer Matrix

B.4 Conjugate Planes, Focal Planes, and Principal Planes

B.4 Conjugate Planes, Focal Planes, and Principal Planes

5.1 A thin lens as a Phase Transformation

- 5.1.1 The thickness function
- 5.1.2 The paraxial approximation
- 5.1.3 The Phase Transformation and its Physical Meaning

5.2 Fourier Transforming Properties of Lenses

- 5.2.1 Input placed against the Lens
- 5.2.2 Input Placed in Front of the Lens
- 5.2.3 Input Placed Behind the Lens
- 5.2.4 Example of an Optical Fourier Transform

5.3 Image Formation: Monochromatic Illumination

- 5.3.1 The impulse Response of a Positive Lens
- 5.3.2 Eliminating Quadratic Phase Factors: The Lens Law
- 5.3.3 The Relation Between Object and Image

5.4 Analysis of Complex Coherent Optical Systems

- 5.4.1 An operator Notation
- 5.4.2 Application of the Operator Approach to Some Optical Systems

B.1 The Domain of Geometrical Optics

Goal: understand the relationship of the geometrical optics and physical optics

Subject of interest is the physical optics formulation of the imaging and spatial filtering and the related concepts in geometrical optics.

Wavelength of the light is always the same but

if the variations or changes of the amplitude and phase of a wavefield take place on spatial scales that are much larger than the wavelength,

then predictions of the geometrical optics become accurate.

Things that take us out of the relm of the geometrical optics:

A sharp edge

A sharply defined aperture

A sharp change of the phase by a significant fraction of 2π over a spatial scales comparable to λ

Example: a grating with $\frac{1}{N} \sim \lambda$ has to be treted by physical optics

a grating $\frac{1}{N} >> \lambda$ can be treated by geometrical optics

The Concept of a Ray

A ray traces the path of power flow in an isotropic medium.

What does this mean?

A monochromatic disturbance traveling in a medium with an index of refraction that varies slowly compared to the wavelength of the disturbance is expressd by:

$$U(\vec{r}) = \underbrace{A(\vec{r})}_{\text{Amplitude}} e^{\overbrace{jk_0 S(\vec{r})}^{\text{Phase}}}$$ where $k_0 = \dfrac{2\pi}{\lambda_0}$ and λ_0 is the free space wavelength.

$S(\vec{r})$ or the *Eikonal Function* contains the effect of the refractive index.

Wavefronts are surfaces defined by $S(\vec{r}) = \text{constant}$

In an isotropic medium: direction of the power flow and direction of the wavevector $\vec{k}$ are both normal to the *wavefronts*. at each point $\vec{r}$.

The geometrical optics limit & Eikonal Equation

$U(\vec{r}) = A(\vec{r})e^{jk_0 S(\vec{r})}$ must satisfy the Helmholtz equation $\left(\nabla^2 + k^2\right)U(\vec{r}) = 0$

1) $\nabla U(\vec{r}) = \nabla\left(A(\vec{r})e^{jk_0 S(\vec{r})}\right) = \nabla A(\vec{r})\left(e^{jk_0 S(\vec{r})}\right) + A(\vec{r})\left(jk_0 \nabla S(\vec{r})\right)e^{jk_0 S(\vec{r})}$

2) $\nabla^2 U(\vec{r}) = (\nabla \bullet \nabla)U(\vec{r}) = \nabla \bullet \left(\nabla A(\vec{r})\left(e^{jk_0 S(\vec{r})}\right) + A(\vec{r})\left(jk_0 \nabla S(\vec{r})\right)e^{jk_0 S(\vec{r})}\right)$

$$\left(\nabla^2 + k^2\right)U(\vec{r}) = \nabla^2 A(\vec{r})\left(e^{jk_0 S(\vec{r})}\right) + \nabla A(\vec{r}) \bullet \left(jk_0 \nabla S(\vec{r})\right)e^{jk_0 S(\vec{r})} + \nabla A(\vec{r}) \bullet \left(jk_0 \nabla S(\vec{r})\right)e^{jk_0 S(\vec{r})}$$

$$+A(\vec{r})\left(jk_0 \nabla^2 S(\vec{r})\right)e^{jk_0 S(\vec{r})} + A(\vec{r})\left(jk_0 \nabla S(\vec{r})\right)^2 e^{jk_0 S(\vec{r})} + n^2 k_0^2 A(\vec{r})e^{jk_0 S(\vec{r})} = 0$$

$$\nabla^2 A(\vec{r}) + k_0^2\left[n^2 - \left|\nabla S(\vec{r})\right|^2\right]A(\vec{r}) + jk_0\left[2\nabla A(\vec{r}) \bullet \nabla S(\vec{r}) + A(\vec{r})\nabla^2 S(\vec{r})\right] = 0$$

We require the both real and imaginary parts be equal to zero independetly.

$$\nabla^2 A(\vec{r}) + k_0^2\left[n^2 - \left|\nabla S(\vec{r})\right|^2\right]A(\vec{r}) = 0 \rightarrow \left|\nabla S(\vec{r})\right|^2 = n^2 + \left(\frac{\lambda_0}{2\pi}\right)\frac{\nabla^2 A(\vec{r})}{A}$$

As $\lambda_0 \rightarrow 0$ we get the so called *Eikonal equation* $\boxed{\left|\nabla S(\vec{r})\right|^2 = n^2}$

Eikonal equation perhaps is the most important equation on the behavior of light under the geometrical optics approximation.

The *Eikonal equation* defines the *wavefront* and therfore the trajectory of the *rays*.

Eikon means image in Greek.

B.2 Refraction, Snell's Law, and the Paraxial Approximation

It can be derived from the Eikonal equation $\left|\nabla S(\vec{r})\right|^2 = n^2(\vec{r})$ that rays in a

a) homogeneous medium (constant n) always travel in straight lines.

b) inhomogeneous medium (varying n) have curved trajectories that depend on the changes of index of refraction.

Waves encountering abrupt changes of n experience abrupt changes in direction of propagation. This is formulated in *Snell's law*:

$n_1 \sin\theta_1 = n_2 \sin\theta_2$

Applying the *paraxial approximation* where $\sin\theta \approx \theta$ and $\cos\theta \approx 1$ the *Snell's law* becomes $n_1\theta_1 = n_2\theta_2$

Defining the *reduced angle* $\widehat{\theta} = n\theta$ we have a simple form for *Snell's law*

$\widehat{\theta}_1 = \widehat{\theta}_2$

B.3 The Ray Transfer Matrix

An equivalent formalism to the operator methods of the physical optics in geometrical optics is the matrix formalism which is valid under paraxial approximation.

To apply the matrix formalism we need to limit ourselves to *meridonal* rays.

Meridonal rays are the rays that are traveling in a single plane containing the z axis.

The transevers axis in the *meridonal* plane is caled y axis by tradition.

In figure a typical ray propagation problem is demonstrated.

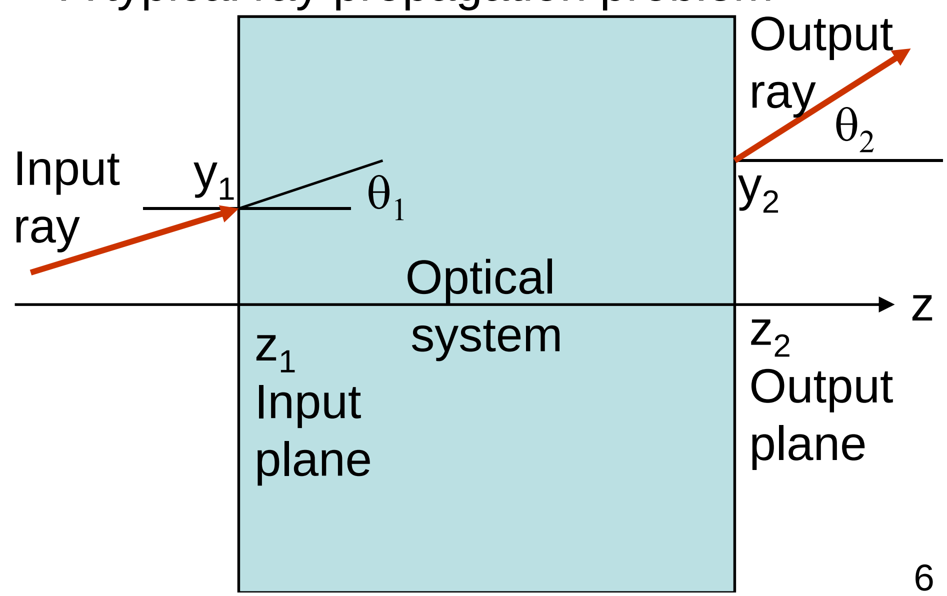
Goal: to determine the position y_2 and angle θ_2 of the output in terms of the position y_1 and angle θ_1 of the input.

Fact: under the paraxial approximation the relationship between (y_2, θ_2) and (y_1, θ_1) are linear and written as:

$$\begin{cases} y_2 = Ay_1 + B\hat{\theta}_1 \\ \hat{\theta}_2 = Cy_1 + D\hat{\theta}_1 \end{cases} \rightarrow \begin{pmatrix} y_2 \\ \hat{\theta}_2 \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} y_1 \\ \hat{\theta}_1 \end{pmatrix}$$

the matrix $\mathbf{M} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is called *ray-transfer* matrix or the *ABCD* matrix.

A typical ray propagation problem



2.2 Spatial frequency and space-frequency localization I

Each Fourier component of a function is a complex exponential function of a unique spatial frequency. Therefore every frequency component extends over the entire (x, y) domain.

So we can't associate a spatial location for a particular spatial frequency.

$$g(x,y)=\iint_{-\infty}^{\infty} G\left(f_X, f_Y\right) e^{j2\pi(f_X x+f_Y y)} df_X df_Y$$

$$G(f_X, f_Y)=\iint_{-\infty}^{\infty} g\left(x,y\right) e^{-j2\pi(f_X x+f_Y y)} dxdy$$

In practice certain portions of an image could contain parallel grid lines at a certain fixed spaing. We tend to say these frequencies are localized to the certain spatial regions of the image.

What is the relationship of the local spatial frequencies or (f_{lX} and f_{lY}) with the spatial frequencies of the Fourier components?

Spatial frequency and space-frequency localization II

Consider a complex valued function: $g(x,y)=a(x,y)e^{j\phi(x,y)}$ where $a(x,y)$ is a positive slowly varying amplitude function and $\phi(x,y)$ is a real phase distribution.

We define the local spatial frequency of the function $g(x,y)=a(x,y)e^{j\phi(x,y)}$:

$$f_{lX}=\frac{1}{2\pi}\frac{\partial}{\partial x}\phi(x,y) \quad \text{and} \quad f_{lY}=\frac{1}{2\pi}\frac{\partial}{\partial y}\phi(x,y) \quad \text{also } f_{lX}=0 \ \& \ f_{lY}=0 \ \text{if} \ g(x,y)=0$$

When <u>the rate of phase change is constant</u> we have a fixed f_l as rate chages then we move to a different f_l. Now for $g(x,y)=e^{j2\pi(f_X x+f_Y y)}$ with $a(x,y)=1$ we obtain

$$f_{lX}=\frac{1}{2\pi}\frac{\partial}{\partial x}2\pi(f_X x+f_Y y)=f_X \quad \text{and} \quad f_{lY}=\frac{1}{2\pi}\frac{\partial}{\partial y}2\pi(f_X x+f_Y y)=f_Y$$

<u>So for the case of a single Fourier component the local spatial frequencies f_l and f are the same and span over the entire (x,y) plane (no localization).</u>

Spatial frequency and space-frequency localization III

Now we consider a space limited version of the quadratic phase exponent or a finite chirp function (chirp function is another name for the infinite-length quadratic phase exponent $e^{j\pi\beta(x^2+y^2)}$):

$$g(x,y) = e^{j\pi\beta(x^2+y^2)} rect\left(\frac{x}{2L_X}\right) rect\left(\frac{y}{2L_Y}\right)$$

$$f_{lX} = rect\left(\frac{x}{2L_X}\right)\frac{1}{2\pi}\frac{\partial}{\partial x}\pi\beta(x^2+y^2) = \beta x \; rect\left(\frac{x}{2L_X}\right)$$

$$f_{lY} = rect\left(\frac{y}{2L_Y}\right)\frac{1}{2\pi}\frac{\partial}{\partial y}\pi\beta(x^2+y^2) = \beta y \; rect\left(\frac{y}{2L_Y}\right)$$

We see now that $f_{lX} = \beta x \; rect\left(\frac{x}{2L_X}\right)$ and $f_{lY} = \beta y \; rect\left(\frac{y}{2L_Y}\right)$ both depend on location on the (x,y) plane within the rectangle of dimensions $2L_X \times 2L_Y$, f_{lX} varies linearly with x and f_{1Y} varies linearly with y.

So for this function and many other functions there is a dependence of local spatial frequenciy on the position in the (x,y) plane.

Local Spatial Frequencies and the Ray-Transfer Matrix

Relationship of the local spatial frequencies f_l and the $ray\text{-}transfer$ matrix:

Under the paraxial approximation the reduced ray angle $\widehat{\theta}$ is related to f_l the local spatial frequencies through

$$f_l = \frac{\theta}{\lambda} = \frac{\widehat{\theta}}{\lambda_0}$$

We see that local spatial frequencies of a coherent optical wavefront correspond to the ray directions of the geometrical optics description of that wavefront.

$$\begin{pmatrix} y_2 \\ \widehat{\theta}_2 \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} y_1 \\ \widehat{\theta}_1 \end{pmatrix} \rightarrow \begin{pmatrix} y_2 \\ f_{l2}\lambda_0 \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} y_1 \\ f_{l1}\lambda_0 \end{pmatrix}$$

So $\mathbf{M}$, the ray-transfer matrix relates the spatial distribution of the local spatial frequencies f_l (or ray directions) at the output to the spatial distribution of the f_l (or ray direction) at the input.

Elementary Ray-Transfer Matrices I

Ray-transfer matix of some important structures

1. Propagation through free space of index n.

The angle of propagation stays unchnged $\theta_2 = \theta_1 = \theta$

while y_2 changes $\tan\theta \approx \theta = \dfrac{y_2 - y_1}{d}$

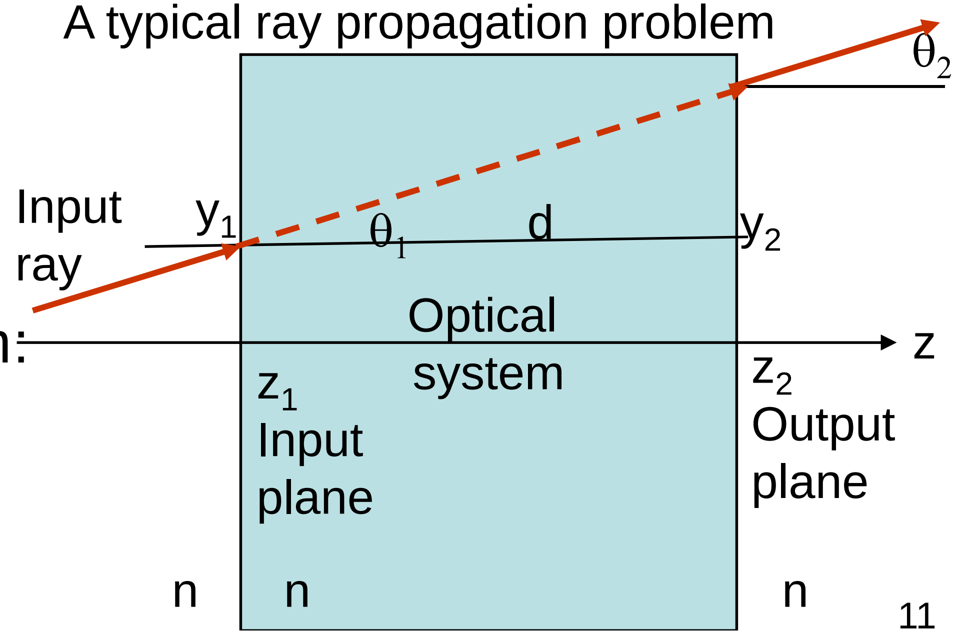
$$\begin{pmatrix} y_2 \\ \hat{\theta}_2 \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} y_1 \\ \hat{\theta}_1 \end{pmatrix} \rightarrow \begin{pmatrix} \theta d + y_1 \\ \hat{\theta} \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} y_1 \\ \hat{\theta} \end{pmatrix}$$

$$\rightarrow \begin{cases} \theta d + y_1 = A y_1 + B\hat{\theta} \\ C y_1 + D\hat{\theta} = \hat{\theta} \end{cases}$$

Therefore a matrix that satisfies the above equation has the form:

$$\mathbf{M} = \begin{pmatrix} 1 & d/n \\ 0 & 1 \end{pmatrix}$$

A typical ray propagation problem



Elementary Ray-Transfer Matrices II

2. Refrction at a planar interface.

$$\begin{pmatrix} y_2 \\ \hat{\theta}_2 \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} y_1 \\ \hat{\theta}_1 \end{pmatrix} \rightarrow \begin{cases} y_2 = Ay_1 + B\hat{\theta}_1 \\ \hat{\theta}_2 = Cy_1 + D\hat{\theta}_1 \end{cases}$$

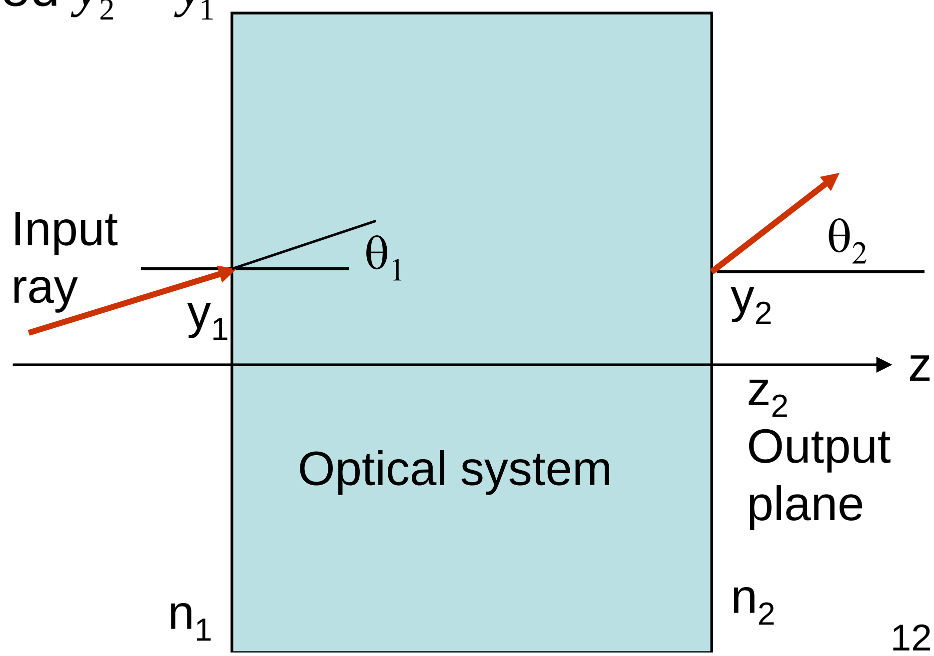
At a planar interfac the angle of propagation change $\theta_2 \neq \theta_1$ but according to the Snell's law $n_2\theta_2 = n_1\theta_1$

Therefore the reduced angle remains unchanged $\hat{\theta}_2 = \hat{\theta}_1$

while position of the ray is unchanged $y_2 = y_1$

Therefore a matrix that satisfies the above equation has the form:

$$\mathbf{M} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

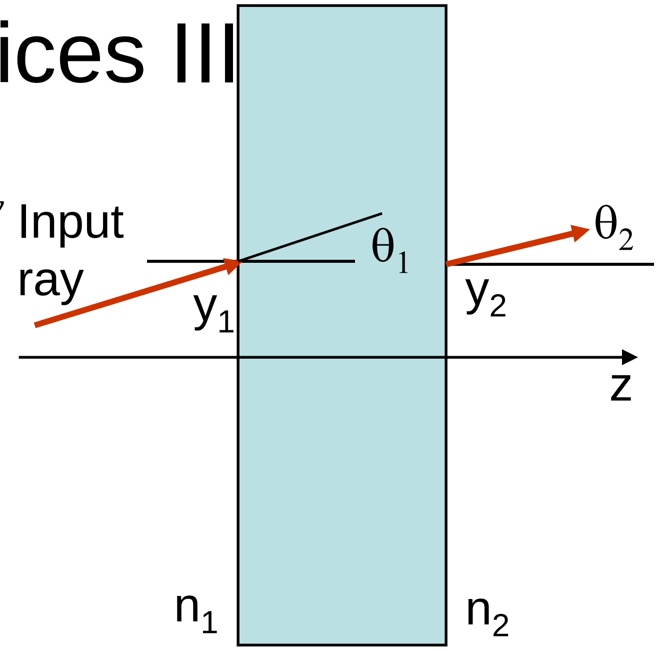


Elementary Ray-Transfer Matrices III

3. Refrction at a spherical interface.

At a spherical interfac the position of ray is not changed $y_1 = y_2 = y$ while the angle of propagation with respect the the normal to the surface, ϕ_2 change according to the Snell's law $n_2\phi_2 = n_1\phi_1$

The relationship of the ϕ to the angle with the optical axis θ is

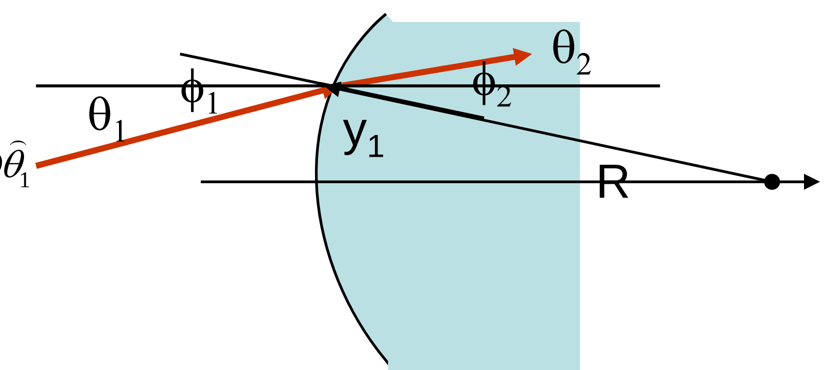


$\phi_1 = \theta_1 + \psi$ where $\psi = \arcsin\frac{y}{R} \approx \frac{y}{R}$ so $\phi_1 = \theta_1 + \frac{y}{R}$ and $\phi_2 = \theta_2 + \frac{y}{R}$

With $n_2\phi_2 = n_1\phi_1 \;\rightarrow\; n_1\theta_1 + n_1\frac{y}{R} = n_2\theta_2 + n_2\frac{y}{R} \;\rightarrow\; \hat{\theta}_1 + n_1\frac{y}{R} = \hat{\theta}_2 + n_2\frac{y}{R}$

$$\hat{\theta}_2 = \hat{\theta}_1 + \frac{(n_1 - n_2)}{R}y$$

$$\begin{pmatrix} y \\ \hat{\theta}_1 + \frac{(n_1 - n_2)}{R}y \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} y \\ \hat{\theta}_1 \end{pmatrix} \rightarrow \begin{cases} y = Ay + B\hat{\theta}_1 \\ \hat{\theta}_1 + \frac{(n_1 - n_2)}{R}y = Cy + D\hat{\theta}_1 \end{cases}$$



Therefore a matrix that satisfies the above equation has the form:

$$\mathbf{M} = \begin{pmatrix} 1 & 0 \\ \frac{(n_1 - n_2)}{R} & 1 \end{pmatrix}$$

A positive value for R satisfies the convex surface encountered from left to right.

A negative value for R satisfies the concave surface encountered from left to right.

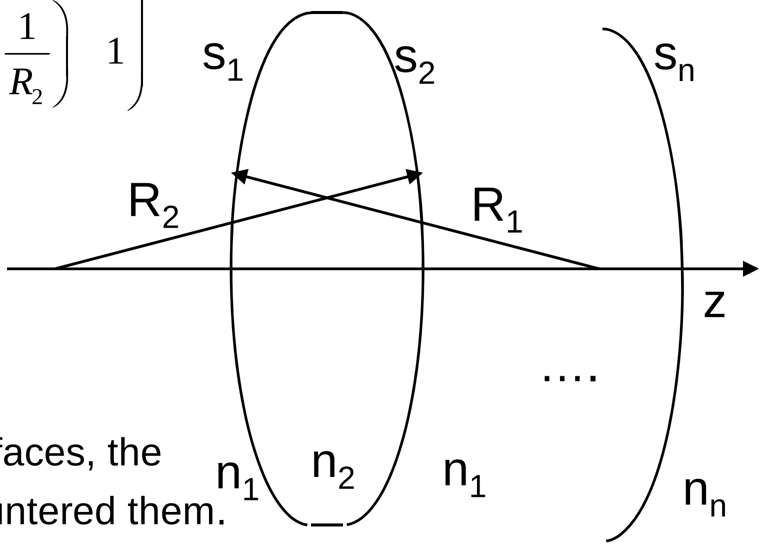
Elementary Ray-Transfer Matrices IV

4. Passage through a thin lens.

A thin lens with index n_2 embedded in a medium of n_1 can be treated by cascading two spherical interfaces. for the second surface roles of n_1 and n_2 are switched. Ray-transfer matix of the

surface 1: $\mathbf{M}_1 = \begin{pmatrix} 1 & 0 \\ \dfrac{(n_1 - n_2)}{R_1} & 1 \end{pmatrix}$ and surface 2: $\mathbf{M}_2 = \begin{pmatrix} 1 & 0 \\ \dfrac{(n_2 - n_1)}{R_2} & 1 \end{pmatrix}$

The total effect of both syrfaces: $\mathbf{M} = \mathbf{M}_2\mathbf{M}_1 = \begin{pmatrix} 1 & 0 \\ -(n_2 - n_1)\left(\dfrac{1}{R_1} - \dfrac{1}{R_2}\right) & 1 \end{pmatrix}$



Defining the focal length of the lens: $\dfrac{1}{f} = \dfrac{n_2 - n_1}{n_1}\left(\dfrac{1}{R_1} - \dfrac{1}{R_2}\right)$

$$\mathbf{M} = \begin{pmatrix} 1 & 0 \\ -\dfrac{n_1}{f} & 1 \end{pmatrix}$$

To use the Ray-transfer matrix for many instruments and surfaces, the matrices should appear in the order which the ray has emcountered them.

$\mathbf{M} = \mathbf{M}_n ... \mathbf{M}_2\mathbf{M}_1$

Since we have used the reduced angles insted of real angles, determinant of all the transfer matrices have become unity.

Cardinal points and planes in Gaussian optics

- The **cardinal points** and the associated **cardinal planes** are a set of special points and planes in an optical system, which help in the analysis of its paraxial properties. The analysis of an optical system using cardinal points is known as **Gaussian optics**, named after Carl Friedrich Gauss.
  – The cardinal points and planes of an optical system include:
  – The **focal points** and **focal planes**
  – The **principal planes** and **principal points**
  – The **surface vertices** (or *vertexes*).
  – The **nodal points**

B.4 Conjugate Planes

Conjugate planes: Two planes or points within an optical system are said to be conjugate planes if intensity distribution across one plane is an image of the intensity distribution across the other plane.

Using $\begin{cases} y_2 = Ay_1 + B\widehat{\theta}_1 \\ \widehat{\theta}_2 = Cy_1 + D\widehat{\theta}_1 \end{cases}$ we can show for the conjugate points on the input and output plane

1) the position of the point y_2 should be independent of the reduced angle $\widehat{\theta}_1$ of a ray throug y_1 therefore $B = 0$.

2) $y_2 = m_t\ y_1$ where m_t is the transverse magnification. Therefore $A = m_t$.

3) the angles of the rays passing through y_2 will be maginfied or demagnified with respect to the angles of the rays arriving from y_1. If $m_\alpha = reduced\ \ angular\ \ magnification$ then $\widehat{\theta}_2 = m_\alpha \widehat{\theta}_1$. Therefore $D = m_\alpha$

And $\mathbf{M} = \begin{pmatrix} m_t & 0 \\ 0 & m_\alpha \end{pmatrix}$

Angle $\left(\cos^{-1}(k_y / k)\right)$ and position y are conjugate Fourier variables.

The similarity (scaling) theorem implies that

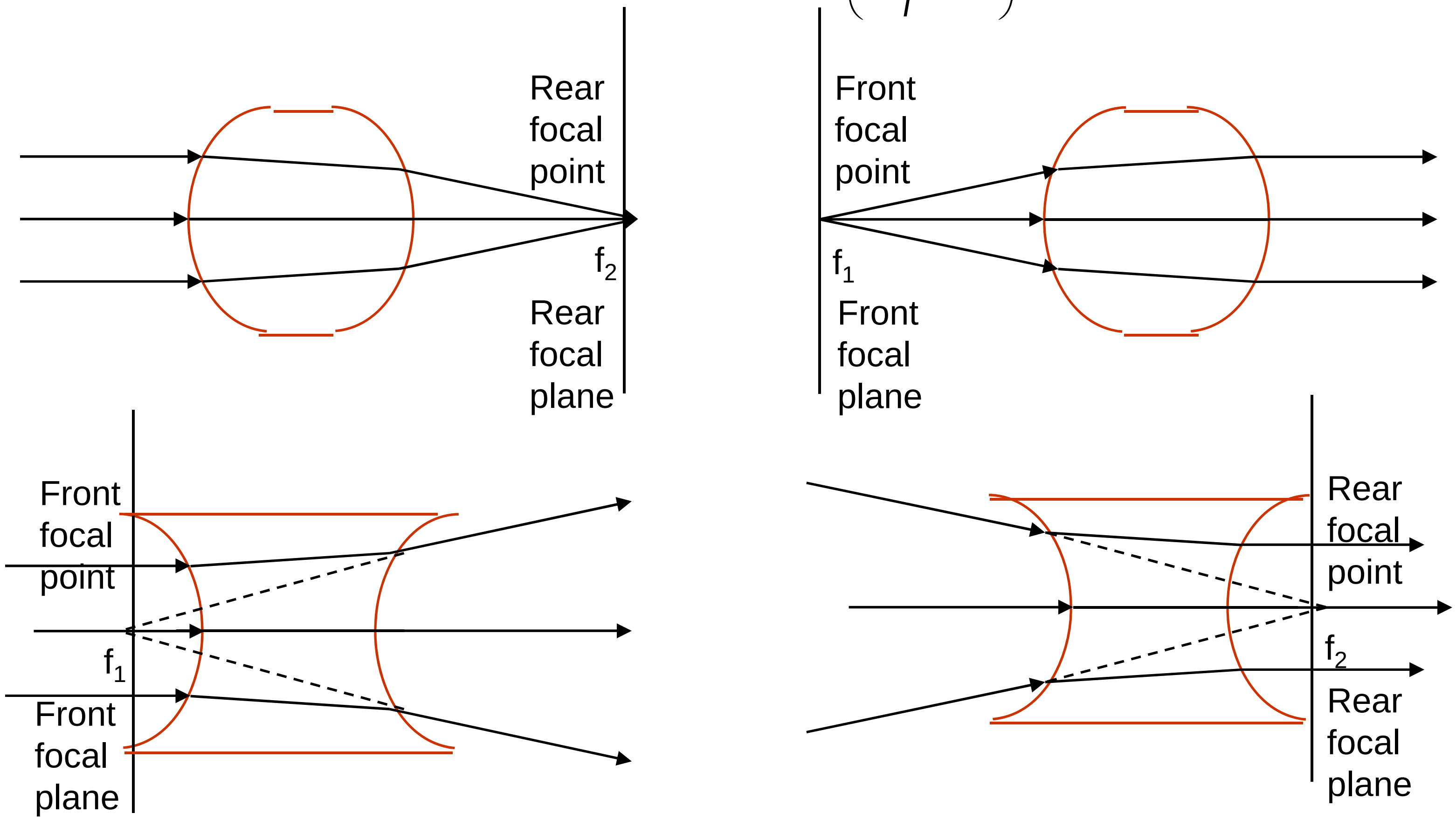
$$y_1\widehat{\theta}_1 = y_2\widehat{\theta}_2 = 1 \rightarrow y_1 / y_2 = \widehat{\theta}_2 / \widehat{\theta}_1 \rightarrow m_t^{-1} = m_\alpha \rightarrow m_t m_\alpha = 1 \quad \rightarrow \mathbf{M} = \begin{pmatrix} m_t & 0 \\ 0 & m_t^{-1} \end{pmatrix}$$

Note m_t and m_α can be both positive and negative but they have to be of the same sign.

Nonparaxial form of the $m_t m_\alpha = 1$ is caled sine condition: $\boxed{n_1 y_1 \sin\theta_1 = n_2 y_2 \sin\theta_2}$

B.4 Focal Planes

For a parallel bundle of rays traveling parallel to the optical axis and entering a lens, there always will exist a point on the optical axis toward which the ray bundle converge (positive lanse) or from which it will appear to diverge (negative lens). If f is the focal length of the lens, then the ray-transfer matrix between the focal planes takes the forrm $\mathbf{M} = \begin{pmatrix} 0 & \dfrac{f}{n_1} \\ -\dfrac{n_1}{f} & 0 \end{pmatrix}$ Exercise: prove this



B.4 Principal Planes

For thin lens $y_2 = y_1$. For thick lens $y_2 \neq y_1$.

We can assume that focusing power of a thick lens is concentrated on a set of planes. We call such planes principal planes (see the figures).

Principal planes are introduced to make the thick lens analysis simple.

In general principal points do not make a flat surface but for small aperture lenses that we are interested in, they can be considered flat.

The principal planes are conjugate of each other with "unit" magnification i.e. $y_1 = y_2$ but $\theta_1 \neq \theta_2$

For thin lenses P_1 and P_2 coincide and become one plane but in general they are two separate planes.

Ray-transfer matrix for propagation of light between two principal planes:

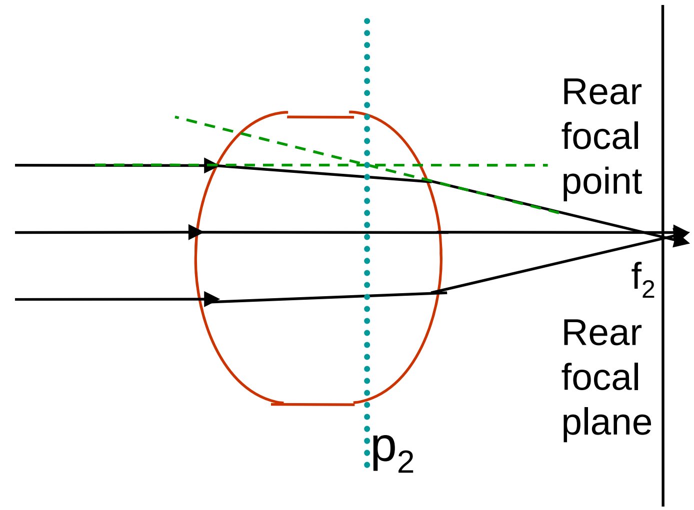
1) If parallel rays arrive at P_1, then system yeilds rays that are converging from the P_2 towards f_2 then $y_1 = y_2$ & $\hat{\theta}_1 = 0$ & $\hat{\theta}_2 = Cy_1 = n(-y_2)/f_2 \rightarrow C = -n/f_2$

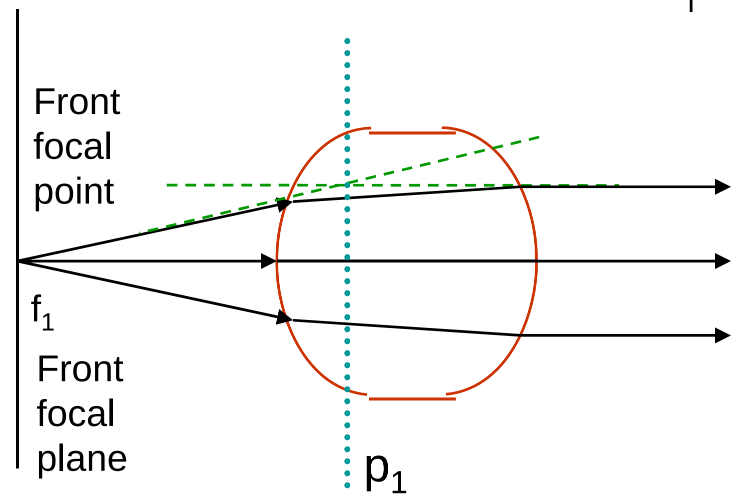
$$\begin{cases} y_1 = Ay_1 + B\hat{\theta}_1 \\ \hat{\theta}_2 = Cy_1 + D\hat{\theta}_1 \end{cases} \rightarrow \begin{pmatrix} 1 & B \\ -\dfrac{n}{f_2} & 1 \end{pmatrix}$$

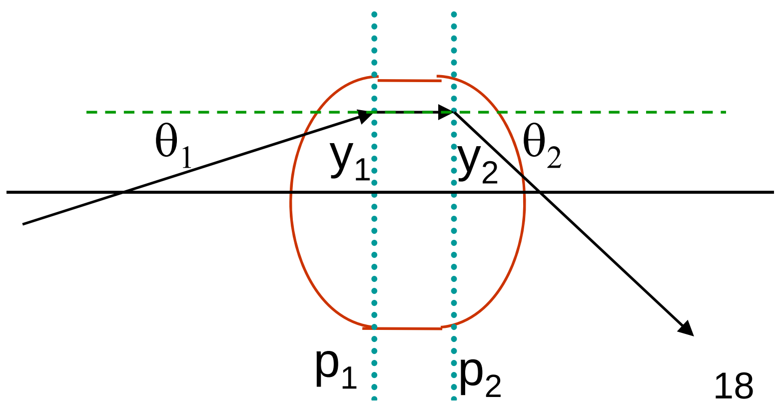
2) If a point source is located on the f_1, then it yields collimated beam leaving P_2. Then $y_1 = y_2$ & $B = 0$ & $\hat{\theta}_2 = 0$ so $C/D = -\hat{\theta}_1/y_1 = -n\theta_1/y_1 = -n/f_1$

since $D = 1$ then $C = -n/f_1$ therefore $\begin{cases} y_1 = Ay_1 + B\hat{\theta}_1 \\ \hat{\theta}_2 = Cy_1 + D\hat{\theta}_1 \end{cases} \rightarrow \begin{pmatrix} 1 & B \\ -\dfrac{n}{f_1} & 1 \end{pmatrix}$

$$C = -\frac{n}{f_1} = -\frac{n}{f_2} \rightarrow f = f_1 = f_2, \text{ and } \mathbf{M} = \begin{pmatrix} 1 & B \\ -\dfrac{n}{f} & 1 \end{pmatrix}$$







B.5 Exit and entrance pupils

Effects of pupils or finite apertures on the ray-transfer.

Apertures are sources of diffraction so they have influence on the image. An optical system may have many limiting apetures.

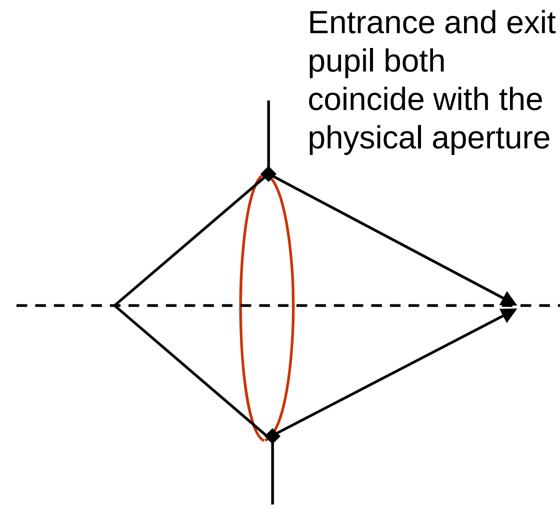
1) enterance pupil of the optical system is the image of the most severely limiting aperture, when viewed from the object space, looking through any optical elements that may proceed the optical system.

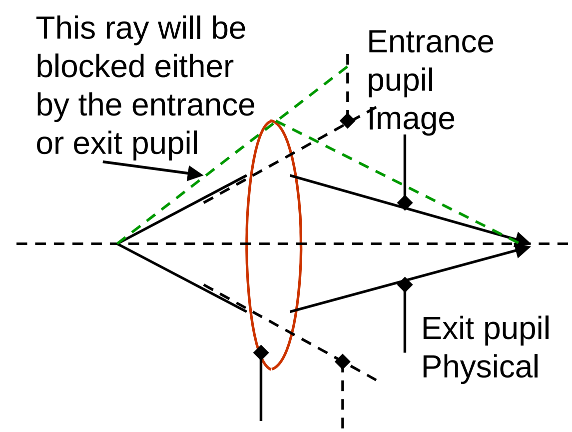
2) exit pupil of the optical system is the image of the most severely limiting physical aperture, looking from the image space through any optical elements that may lie between the aperture and the image plane.

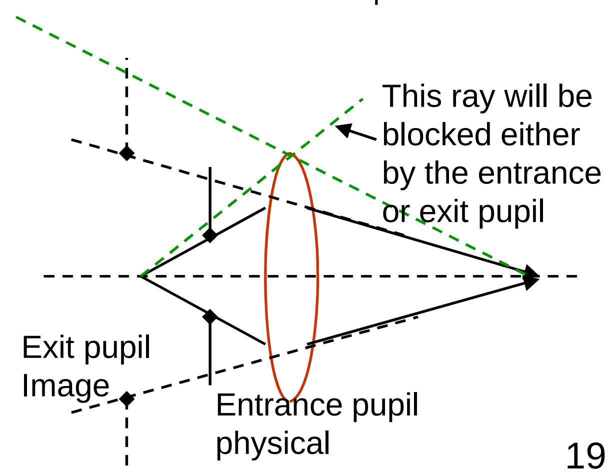
For more complicated optical systems with many lenses and apertures using ray tracing we find the most severely affecting aperture and then finding its image viewd from the object space and image space we determine the enterance and exit pupils of the system.

For an abberation-free system image of a point-source is a perfect point. Introducing the exit pupil, diffraction effects are observed on the image. We can use the diffraction formula to calculate the spatial distribution of the image of a point source.

The distance from the exit pupil to the image plane is the distance z appearing in the diffraction formula.



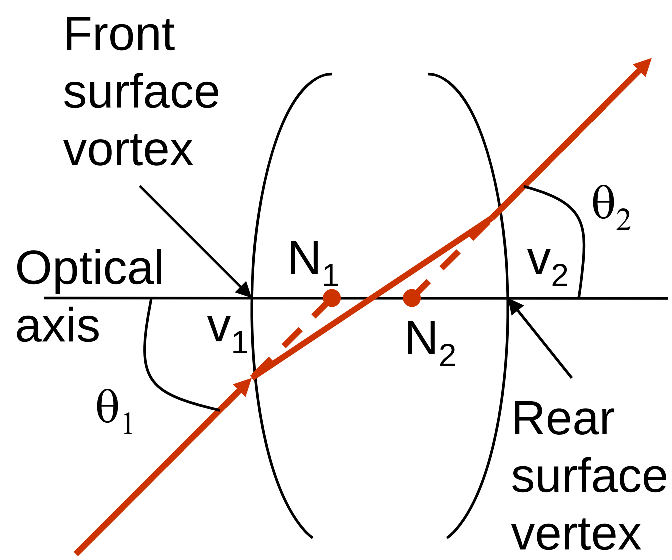




The Surface Vertices and Nodal points

- The surface vertices are the points where each surface crosses the optical axis. They are important primarily because they are the physically measurable parameters for the position of the optical elements, and so the positions of the other cardinal points must be known with respect to the vertices to describe the physical system.

- The front and rear nodal points have the property that a ray that passes through one of them will also pass through the other, and with the same angle with respect to the optical axis. The nodal points therefore do for angles what the principal planes do for transverse distance. If the medium on both sides of the optical system is the same (e.g. air), then the front and rear nodal points coincide with the front and rear principal planes, respectively.
- **Exercise: Find the transfer matrix for the nodal points**



5.1 A thin lens as a Phase Transformation

lens: an optically dense material with $n > 1$

Speed of light is less than c in a lens.

Thin lens: a ray arriving at coordinates (x, y) on one face, exits at approximately the same coordinates on the opposite face or $y_2 = y_1$.

A thin lens simply delays an incident wavefront by an amount proportional to the thickness of the lens at each point.

Phase delay means change in the wavefront shape, therefore change in direction of parpagation vector. We define:

$\Delta_0 =$ Maximum thickness

$\Delta(x, y) =$ Thickness at coordinates x, y

Total phase delay suffered by the wave at (x, y) passing through the lens:

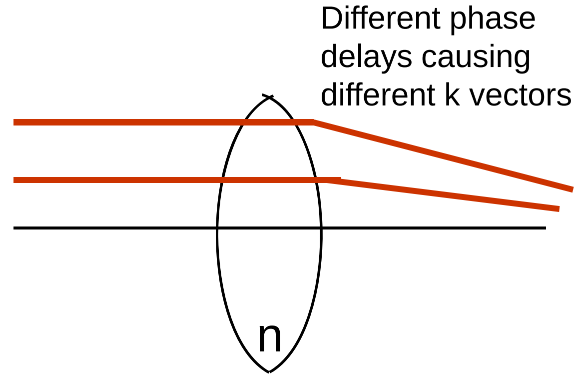
$$\phi(x, y) = k\Delta_0 - k\Delta(x, y) + kn\Delta(x, y)$$

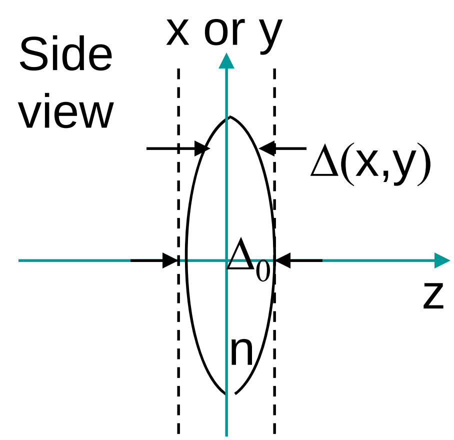
The lens may be represented by a multiplicative phase transformation of the form:

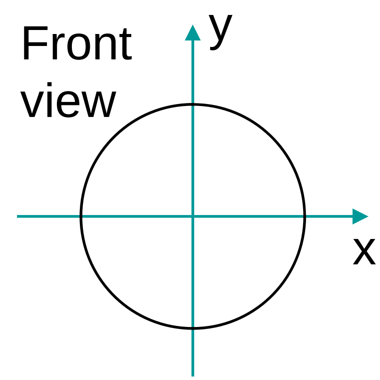
$$t_l = e^{jk\Delta_0} e^{[jk(n-1)\Delta(x,y)]}$$

$$\underbrace{U_l^{'}(x, y)}_{\text{Complex field across a plane imediately behind the lens}} = \underbrace{t_l(x, y)}_{\text{Phase delay caused by the lens}} \underbrace{U_l(x, y)}_{\text{Complex field incident on a plane imediately in front of the lens}}$$

If we know the mathematical form of the thickness function $\Delta(x, y)$ then we can calculate $U_l^{'}(x, y)$ and unederstand the effects of the lens.







5.1.1 The thickness function

Goal: define the thickness function $\Delta(x,y)$ for a lens.

Sign convention:

Rays travel from left to right.

1) Radius of convex surfaces encountered by the lens are positive.

2) Radius of concave surfaces encountered by the lens are negative.

To find $\Delta(x,y)$ we split the lens to three parts and write the total thicknes function as:

$\Delta(x,y)=\Delta_1(x,y)+\Delta_2(x,y)+\Delta_3(x,y)$ with $r^2=x^2+y^2$

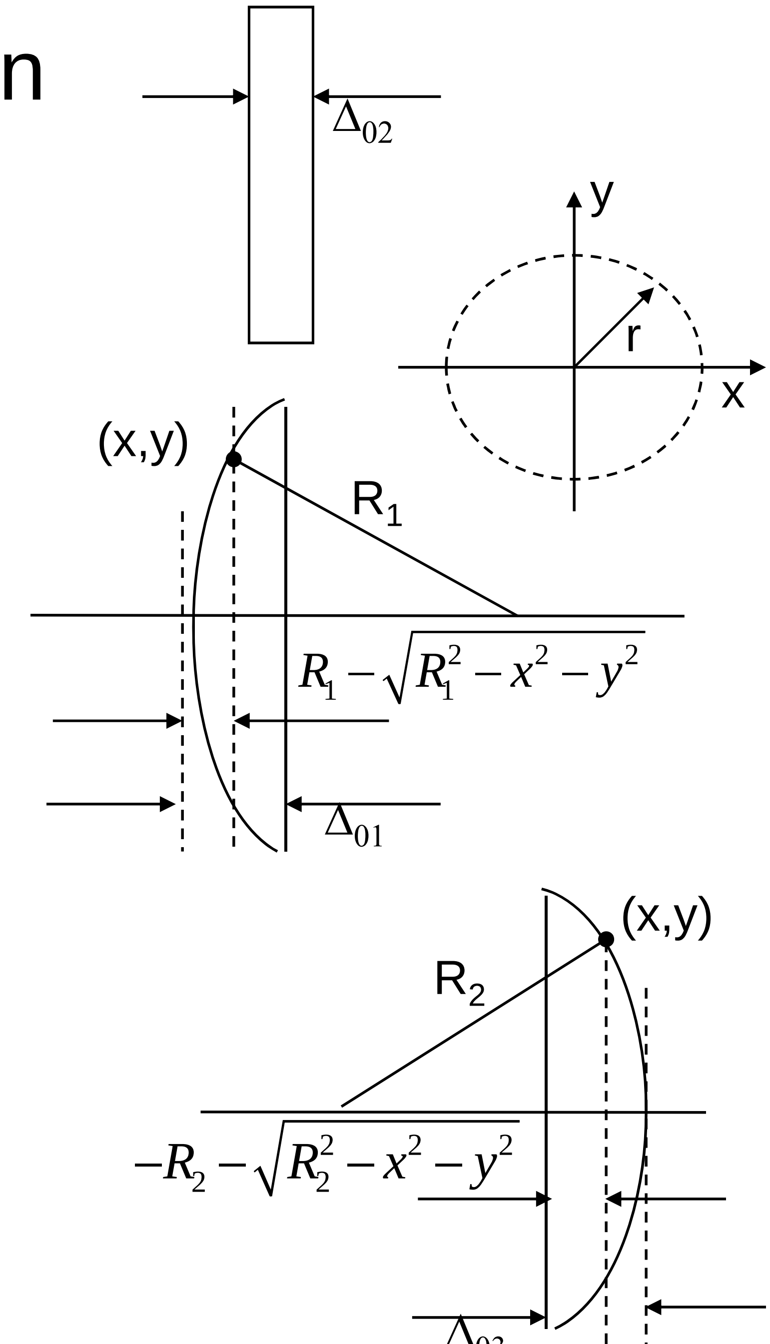
$$\Delta_1(x,y)=\Delta_{01}-\left(R_1-\sqrt{R_1^2-r^2}\right)=\Delta_{01}-R_1\left(1-\sqrt{1-\frac{x^2+y^2}{R_1^2}}\right)$$

$$\Delta_2(x,y)=\Delta_{02}$$

$$\Delta_3(x,y)=\Delta_{03}-\left(-R_2-\sqrt{R_2^2-r^2}\right)=\Delta_{03}+R_2\left(1-\sqrt{1-\frac{x^2+y^2}{R_2^2}}\right)$$

$$\Delta(x,y)=\Delta_0-R_1\left(1-\sqrt{1-\frac{x^2+y^2}{R_1^2}}\right)+R_2\left(1-\sqrt{1-\frac{x^2+y^2}{R_2^2}}\right)$$

Where $\Delta_0=\Delta_{01}+\Delta_{02}+\Delta_{03}$



5.1.2 The paraxial approximation

Goal: to simplify the thickness function for the cases that are restricted to portions of wave near the lens axis.

That means values of x and y are sufficiently small to write:

$$\sqrt{1-\frac{x^2+y^2}{R_1^2}} \approx 1-\frac{x^2+y^2}{2R_1^2} \text{ and } \sqrt{1-\frac{x^2+y^2}{R_2^2}} \approx 1-\frac{x^2+y^2}{2R_2^2}$$

Then we have:

$$\Delta(x,y) = \Delta_0 - R_1\left(1-\sqrt{1-\frac{x^2+y^2}{R_1^2}}\right) - R_2\left(\sqrt{1-\frac{x^2-y^2}{R_2^2}}-1\right)$$

$$\Delta(x,y) \approx \Delta_0 - R_1\left(1-\left(1-\frac{x^2+y^2}{2R_1^2}\right)\right) - R_2\left(\left(1-\frac{x^2+y^2}{2R_2^2}\right)-1\right)$$

$$\Delta(x,y) \approx \Delta_0 - \frac{x^2+y^2}{2}\left(\frac{1}{R_1}-\frac{1}{R_2}\right)$$

5.1.3 The Phase Transformation and its physical meaning

The lens representation as a multiplicative phase transformation:

$$t_l\left(x,y\right)=e^{jk\Delta_0}e^{\left[jk(n-1)\Delta(x,y)\right]}$$

$$U_l^{'}\left(x,y\right)=t_l\left(x,y\right)U_l(x,y)$$

Substituting $\Delta\left(x,y\right)\approx\Delta_0-\frac{x^2+y^2}{2}\left(\frac{1}{R_1}-\frac{1}{R_2}\right)$ in $t_l\left(x,y\right)$ we get:

$$t_l\left(x,y\right)=e^{jk\Delta_0}e^{\left[jk(n-1)\left(\Delta_0-\frac{x^2+y^2}{2}\left(\frac{1}{R_1}-\frac{1}{R_2}\right)\right)\right]}=e^{jkn\Delta_0}e^{\left[-jk\frac{x^2+y^2}{2}(n-1)\left(\frac{1}{R_1}-\frac{1}{R_2}\right)\right]}$$

We combine physical properties of the lens in a single parameter f

$\frac{1}{f}=(n-1)\left(\frac{1}{R_1}-\frac{1}{R_2}\right)$ then $t_l\left(x,y\right)=e^{jkn\Delta_0}e^{\left[-j\frac{k}{2f}\left(x^2+y^2\right)\right]}$.

If we neglect the constant phase factor, the basic representation of the effects of a thin lens on the incident distribution as a pahse transformation factor can be written as:

$$t_l\left(x,y\right)=e^{\left[-j\frac{k}{2f}\left(x^2+y^2\right)\right]}$$

We have not acounted for the finite extent of the lens here.

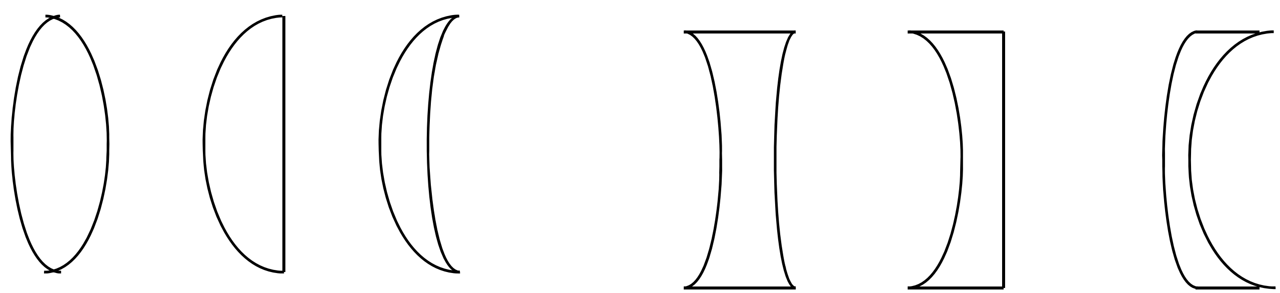
Lens varieties

If we follow the above sign covention, the equation for t_1 represents all six kinds of lens shown in Fig.

Exercise:

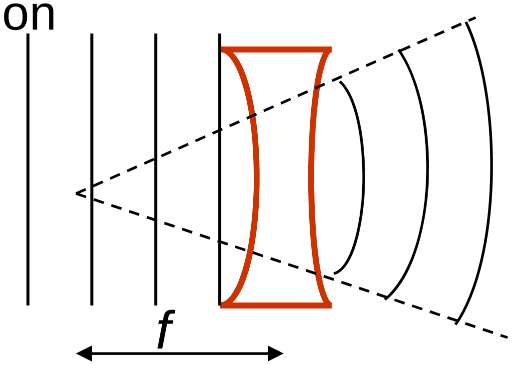
a) show that focal length of a double-convex, plano-convex, and positive meniscus lens is positive.

b) show that focal length of a double-concave, plano-concave and negative meniscus lens is negative.



Double-convex plano-convex Positive meniscus Double-concave plano-concave Positive meniscus

Physical meaning of lens transformation

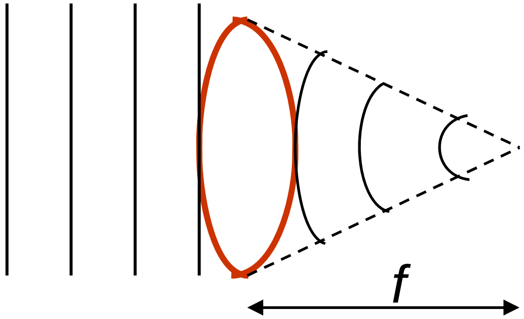


Goal: understand physical meaning of the lens transformation

Illumination: normally incident, unit-amplitude plane wave

The field distribution in front of the lens $U_l(x,y)=1$ then

$$\begin{cases} U_l^{'}(x,y)=t_l(x,y)U_l(x,y) \\ U_l(x,y)=1 \\ t_l(x,y)=e^{\left[-j\frac{k}{2f}\left(x^2+y^2\right)\right]} \end{cases} \Rightarrow \underbrace{U_l^{'}(x,y)=e^{\left[-j\frac{k}{2f}\left(x^2+y^2\right)\right]}}_{\text{A quadratic approximation to a spherical wave}}$$



For $f>0$ the spherical wave is converging towards a point on the axis of lens at a distance f behind the lens.

For $f<0$ the spherical wave is diverging from a point on the axis of lens at a distance f in front of the lens.

Effect of a lens composed of spherical surfaces under the paraxial approximation: plane wavefront $\rightarrow$ spherical wavefront.

Aberrations show on departures from the paraxial regime.

5.2 Fourier Transforming Properties of Lenses

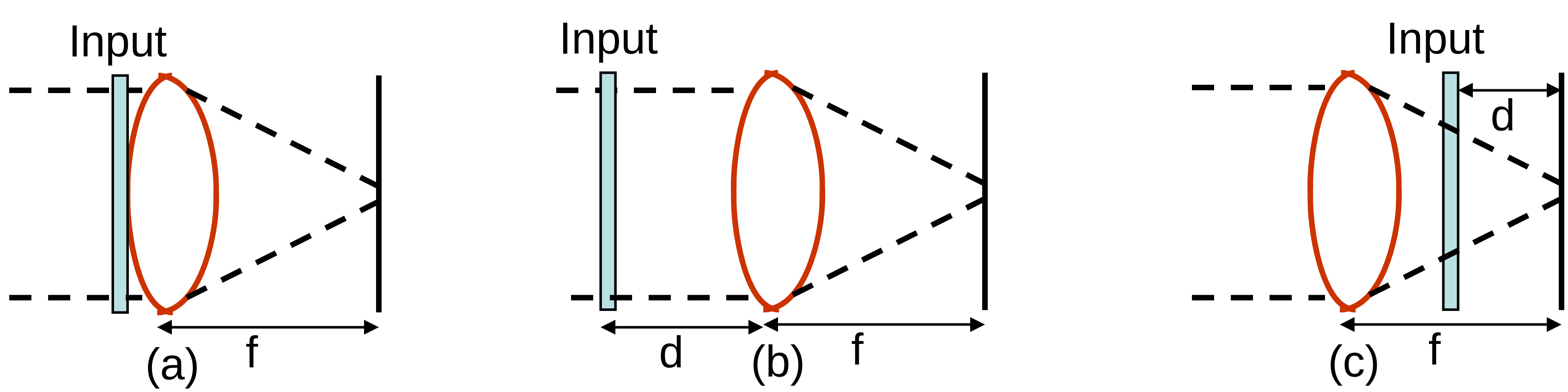
A converging lens performs two-dimensional Fourier transformation which is a very complicated analog operation.

Coherent optical systems: systems with monochromatic illumination that are linear in complex amplitude and the distribution of light amplitude across a particular plane behind the lens is of interest (for example back focal plane).

Input transparencies: a device with amplitude transmittance proportional to the input function that represents the information to be Fourier-transformed.

Input transparencies may also be refered to as object and may be produced by reflection.

Here are several geometries for performing Fourier transform operation with positive lense:



5.2.1 Input placed against the Lens I

Input: a planar transparency with $t_A(x,y)$ located imediately in front of the lens

Fourier transformer: a converging lens with focal length f

Illumination: a uniform (across the transparency ξ,η) normally incident monochromatic plane wave of amplitude A.

Disturbance incident on the lens: $U_l = At_A(x,y)$

Pupil fuunction (for finite extent of the lens): $P(x,y)=\begin{cases}1 & \text{inside the lens aperture}\\ 0 & \text{otherwise}\end{cases}$

Note: here since lens is very close to aperture $(\xi,\eta)\rightarrow(x,y)$

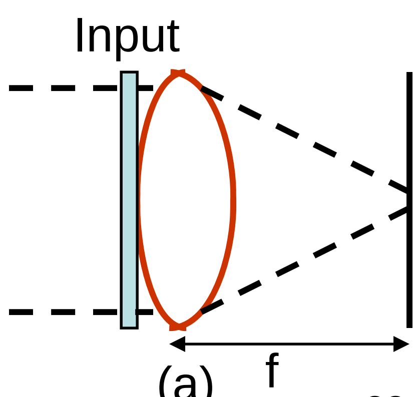
The amplitude distribution behind the lens:

$$U_l'(x,y) = \underbrace{U_l(x,y)}_{\text{Disturbance at the lens}} \underbrace{P(x,y)}_{\text{Lens aperture}} \underbrace{e^{-j\frac{k}{2f}\left(x^2+y^2\right)}}_{\text{Effect of the lens on the phase}}$$

Disturbance on the back focal plane of the lens: $U_f(u,V)$

Using the Fresnel diffraction formula from chapter 4:

$$U(x,y)=\frac{e^{jkz}}{j\lambda z}e^{j\frac{k}{2z}(x^2+y^2)}\int\int_{-\infty}^{\infty}\left\{U(\xi,\eta)e^{j\frac{k}{2z}\left(\xi^2+\eta^2\right)}\right\}e^{-j\frac{2\pi}{\lambda z}(x\xi+y\eta)}d\xi d\eta$$



(a)

5.2.1 Input placed against the Lens II

with $z = f \;\rightarrow\; U_f(u,V) = \dfrac{e^{jkf} e^{j\frac{k}{2f}(u^2+V^2)}}{j\lambda f} \int\int_{-\infty}^{\infty} U_l^{'}(x,y) e^{j\frac{k}{2f}\left(x^2+y^2\right)} e^{-j\frac{2\pi}{\lambda f}(xu+yV)} dxdy$

dropping the constant pahse factor e^{jkf} and substituting

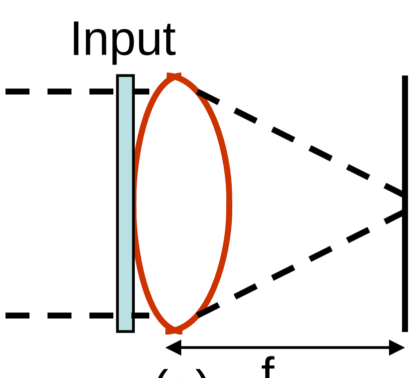
$U_l^{'}(x,y) = U_l(x,y)t_l(x,y) = U_l(x,y)e^{\left[-j\frac{k}{2f}\left(x^2+y^2\right)\right]}$ we get:

$$U_f(u,V) = \frac{e^{j\frac{k}{2f}(u^2+V^2)}}{j\lambda f} \int\int_{-\infty}^{\infty} U_l(x,y)P(x,y) \underbrace{e^{-j\frac{k}{2f}\left(x^2+y^2\right)} e^{j\frac{k}{2f}\left(x^2+y^2\right)}}_{1} e^{-j\frac{2\pi}{\lambda f}(xu+yV)} dxdy$$

$$U_f(u,V) = \frac{e^{j\frac{k}{2f}(u^2+V^2)}}{j\lambda f} \int\int_{-\infty}^{\infty} U_l(x,y)P(x,y) e^{-j\frac{2\pi}{\lambda f}(xu+yV)} dxdy$$

If the physical extent of the input is smaller than the lens the factor $P(x,y)$ may be dropped.

$$U_f(u,V) = \frac{e^{j\frac{k}{2f}(u^2+V^2)}}{j\lambda f} \int\int_{-\infty}^{\infty} U_l(x,y) e^{-j\frac{2\pi}{\lambda f}(xu+yV)} dxdy$$



(a)

5.2.1 Input placed against the Lens III

$$\overbrace{U_f\left(u,V\right)}^{\text{Focal-plane amplitude distribution at coordinates } (u,V)\text{i}} = \overbrace{\frac{e^{j\frac{k}{2f}\left(u^2+V^2\right)}}{j\lambda f}}^{\text{some phase factors}} \overbrace{\int\int_{-\infty}^{\infty} U_l\left(x,y\right) e^{-j\frac{2\pi}{\lambda f}\left(xu+yV\right)} dxdy}^{\text{Fourier transform of the input amplitude transmittance at frequencies } f_X=u/\lambda f \text{ and } f_Y=v/\lambda f}$$

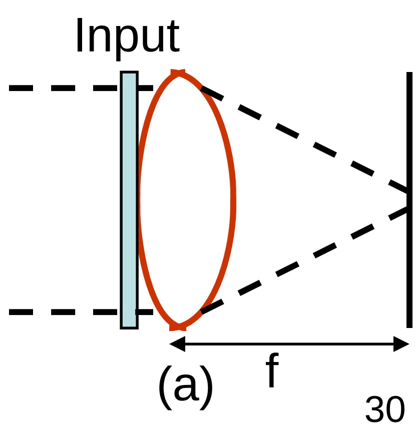
Conclusion: the complex amplitude of the field in the focal plane of the lens is the Fraunhofer diffraction pattern of the field incident on the lens.

Note: here the distance from lens to the observation plane is only f rather than a distance that satisfies the Fraunhofer diffraction criteria.

For the cases that only intensity matters (e.g. photography) these two are the same but when we need to pass the focal-plane amplitude distribution from one lens to another optical system (maybe another lens) then we need the phase information as well so the complete U_f is required.

$$I_f\left(u,V\right) = \frac{A^2}{\lambda^2 f^2}\left|\int\int_{-\infty}^{\infty} t_A\left(x,y\right) e^{-j\frac{2\pi}{\lambda f}\left(xu+yV\right)} dxdy\right|^2$$

Measuring $I_f\left(u,V\right)$, the intensity distribution on the focal plane yields power spectrum or energy spectrum of the input.

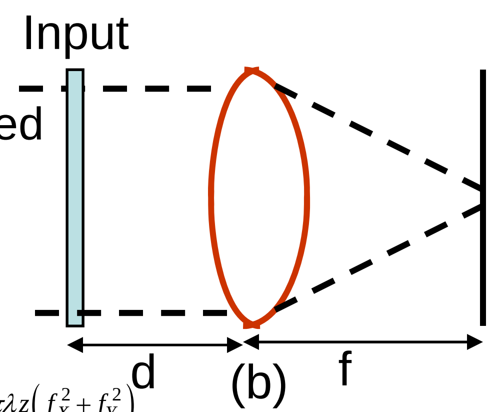


5.2.2 Input Placed in Front of the Lens I

Input: a planar transparency with $t_A(\xi,\eta)$ located in front of the lens at a distance d.

Fourier transformer: a converging lens with focal length f

Illumination: a uniform (across the transparency) normally incident monochromatic plane wave of amplitude A.



$F_0(f_X, f_Y) = \mathcal{F}\{At_A(\xi,\eta)\}$ Fourier spectrum of the light transmitted by the input transparency

$F_l(f_X, f_Y)$ Fourier spectrum of the light incident on the lens.

$$\underbrace{H(f_X, f_Y)}_{\text{Transfer Function of the propagation}} = \begin{cases} e^{-j2\pi\frac{z}{\lambda}\sqrt{1-f_X^2+f_Y^2}} & \sqrt{f_X^2+f_Y^2} < \frac{1}{\lambda} \\ 0 & \text{otherwise} \end{cases} \rightarrow \underbrace{H(f_X, f_Y) = e^{jkz}e^{-j\pi\lambda z\left(f_X^2+f_Y^2\right)}}_{\text{Fresnel approximation for the transfer function of the propagation}}$$

$$F_l(f_X, f_Y) = \mathcal{F}\{U_l\} = \mathcal{F}\left\{\underbrace{At_A(\xi,\eta)}_{\text{Input}}\right\}\underbrace{H(f_X, f_Y)}_{\text{propagation}} = F_0(f_X, f_Y)e^{jkz}e^{-j\pi\lambda d\left(f_X^2+f_Y^2\right)}$$

$F_l(f_X, f_Y) = F_0(f_X, f_Y)e^{-j\pi\lambda d\left(f_X^2+f_Y^2\right)}$ dropped the constant phase

$$\underbrace{U(x,y) = \frac{e^{jkz}}{j\lambda z}e^{j\frac{k}{2z}(x^2+y^2)}\int\int_{-\infty}^{\infty}\left\{U(\xi,\eta)e^{j\frac{k}{2z}\left(\xi^2+\eta^2\right)}\right\}e^{-j\frac{2\pi}{\lambda z}(x\xi+y\eta)}d\xi d\eta}_{\text{Fresnel approximation}}$$

5.2.2 Input Placed in Front of the Lens II

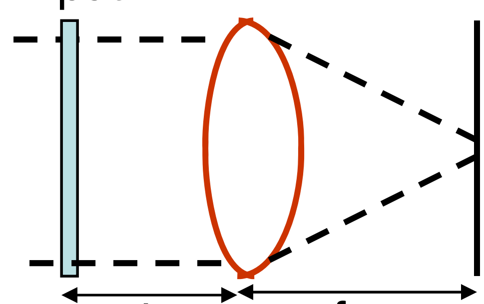
Pupil fuunction (for finite extent of the lens): $P(x,y)=\begin{cases}1 & \text{inside the lens aperture}\\ 0 & \text{otherwise}\end{cases}$

For now we ignore finite extent of the lens.

The field distribution at the focal plane with replacing variables $\xi \to x,\ \eta \to y,$ $x \to u,\ y \to V$ in the Fresnel approximation:

$$U_f(u,V)=\frac{e^{j\frac{k}{2f}(u^2+V^2)}}{j\lambda f}\underbrace{\int\int_{-\infty}^{\infty} U_l(x,y)P(x,y)e^{-j\frac{2\pi}{\lambda f}(xu+yV)}dxdy}_{F_l(f_X,f_Y)\ \text{with}\ f_X=u/\lambda f\ \text{and}\ f_Y=v/\lambda f\ \ P(x,y)=1}=\frac{e^{j\frac{k}{2f}(u^2+V^2)}}{j\lambda f}F_l(f_X,f_Y)$$

Substitute $F_l(f_X,f_Y)=F_0(f_X,f_Y)e^{-j\pi\lambda d\left(f_X{}^2+f_Y{}^2\right)}$ into $U_f(u,V)$:



$$U_f(u,V)=\frac{e^{j\frac{k}{2f}(u^2+V^2)}}{j\lambda f}F_0(f_X,f_Y)e^{-j\pi\lambda d\left(f_X{}^2+f_Y{}^2\right)}\Bigg|_{f_X=u/\lambda f\ \text{and}\ f_Y=v/\lambda f}$$

$$\underbrace{U_f(u,V)}_{\text{Amplitude and phase of the light at }(u,V)}=\underbrace{\frac{e^{j\frac{k}{2f}\left(1-\frac{d}{f}\right)(u^2+V^2)}}{j\lambda f}}_{\text{Quadratic phase factor}}\underbrace{\int\int_{-\infty}^{\infty} t_A(\xi,\eta)e^{-j\frac{2\pi}{\lambda f}(\xi u+\eta V)}d\xi d\eta}_{\text{Amplitude and phase of the input spectrum at frequencies } f_X=u/\lambda f\ \text{and}\ f_Y=v/\lambda f}$$

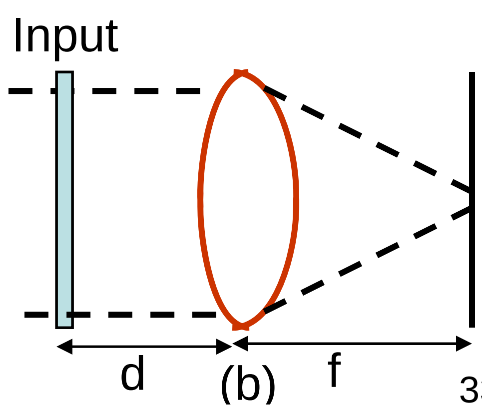
5.2.2 Input Placed in Front of the Lens III

For special case $d = f$

The quadratic phase factor $e^{j\frac{k}{2f}\left(1-\frac{d}{f}\right)(u^2+V^2)} = 1$

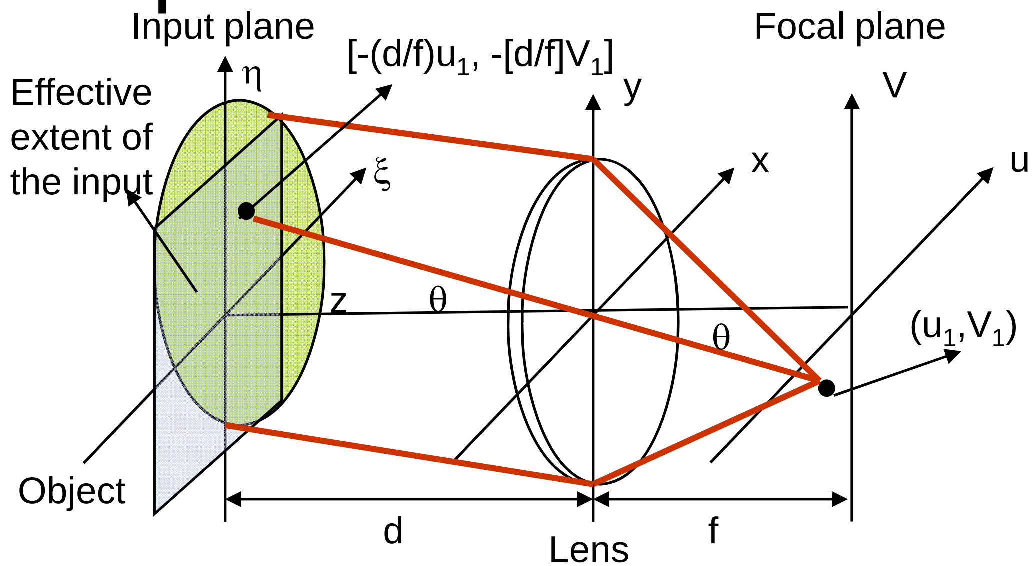
$$\underbrace{U_f(u,V)}_{\substack{\text{Amplitude and} \\ \text{phase of the light} \\ \text{at } (u,V)}} = \frac{1}{j\lambda f} \underbrace{\int\int_{-\infty}^{\infty} t_A(\xi,\eta) e^{-j\frac{2\pi}{\lambda f}(\xi u+\eta V)} d\xi d\eta}_{\substack{\text{Amplitude and phase of the input spectrum} \\ \text{at frequencies } f_X = u/\lambda f \text{ and } f_Y = v/\lambda f}}$$

Conclusion: when the input is placed in front focal plane of the lens, the phase curvature dissapears, and $U_f(u,V)$ is exactly a Fourier transformation of the input transparency $t_A(\xi,\eta)$



5.2.2 Vignetting: limitation of the effective input by the finite lens aperture

Goal: including effect of the finite extent of the lens using geometrical optical approximation. This is valid if d is small enough so that the input is located deep within the Fresnel approximation distance of the lens.



light amplitude at $(u_{1,}V_1) = \sum$ rays with direction cosines $\xi \approx u_1 / f, \eta \approx V_1 / f$

Not all of the rays coming from iput plane at these angles can pas through the lens. Only the ones within the geomtrical shadow of the lens on the input plane meet the condition of apssing through the lens.

U_f at (u,V) can be found from the Fourier transform of that portion of the input subtended by the projected pupil function at angle θ, centered at coordinates $\left[\xi = d(u/f), \eta = d(V/f)\right]$

$$U_f(u,V) = \frac{Ae^{\left[j\frac{k}{2f}\left(1-\frac{d}{f}\right)\left(u^2+V^2\right)\right]}}{j\lambda f}\int\int_{-\infty}^{\infty} t_A(\xi,\eta)P\left(\xi+\frac{d}{f}u, \eta+\frac{d}{f}V\right)e^{-j\frac{2\pi}{\lambda f}(\xi u+\eta V)}d\xi d\eta$$

5.2.2 Effect of Vignetting on practical design issues

- Vignetting is the limitation for the input by the finite lens aperture.
- For a simple Fourier transforming system, vignetting of the input space is minimized if
  - Input is placed close to the lens
  - Lens aperture is much larger than the input transparency
- If we are interested only in the Fourier transform of the input object, we should place it right against the lens to minimize the vignetting.
- On the other hand if the input transparency is located at the front focal plane, then the transform relation is not altered by the quadratic phase factor and that may simplify the problem.

5.2.3 Input Placed Behind the Lens I

Input: a planar transparency with $t_A(x, y)$ located in front of the rare focal plane of the lens at a distance d.

Fourier transformer: a converging lens with focal length f

Illumination: a uniform (across the transparency) normally incident monochromatic plane wave of amplitude A.

The incident wavefront on the input $t_A(x, y)$, is a spherical wave converging towards the back focal plane of the lens.

An approxiamtion based on the following facts:

1) Linear dimension of the cirular bundle at the lens is reduced by a factor of d/f at the input $t_A(x, y)$

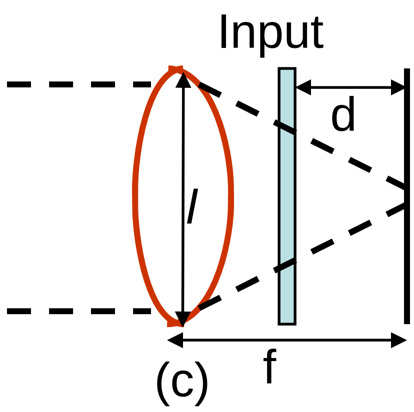
2) Energy has been conserved

Amplitude of the spherical wave impinging on the input: Af/d

Illuminated area on the input: ld/f l is the diameter of the lens

The limitted illumination of the input can be presented by a pupil function at the input plane $P\left[\xi(f/d), \eta(f/d)\right]$

Effective aperture of the system: intersection of the input aperture with the projected pupil function of the lens on the input plane.



5.2.3 Input Placed Behind the Lens II

Using the paraxial approximation for the amplitude of a transmitted wave by a spherical lens is: $U_l'(x,y)=U_l(x,y)P(x,y)e^{-j\frac{k}{2f}(x^2+y^2)}$

we write the amplitude transmitted at the input plane as:

$$U_0(\xi,\eta)=\left\{\frac{Af}{d}P\left[\xi\left(\frac{f}{d}\right),\eta\left(\frac{f}{d}\right)\right]e^{-j\frac{k}{2d}(\xi^2+\eta^2)}\right\}t_A(\xi,\eta)$$

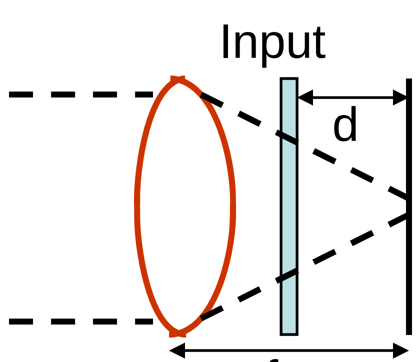
Assuming the Fresnel approx. for the propagation from the input plane to the focal plane:

$$U(x,y)=\frac{e^{jkz}}{j\lambda z}e^{j\frac{k}{2z}(x^2+y^2)}\int\int_{-\infty}^{\infty}\left\{U(\xi,\eta)e^{j\frac{k}{2z}(\xi^2+\eta^2)}\right\}e^{-j\frac{2\pi}{\lambda z}(x\xi+y\eta)}d\xi d\eta \quad \text{with } x\to u \ \&\ y\to V \ \&\ z=d$$

The amplitude at the focal plane becomes:

$$U_f(u,V)=\frac{e^{j\frac{k}{2d}(u^2+V^2)}}{j\lambda d}\frac{Af}{d}\int\int_{-\infty}^{\infty}\left\{t_A(\xi,\eta)P\left[\xi(\frac{f}{d}),\eta\left(\frac{f}{d}\right)\right]e^{-j\frac{k}{2d}(\xi^2+\eta^2)}e^{j\frac{k}{2d}(\xi^2+\eta^2)}\right\}e^{-j\frac{2\pi}{\lambda d}(u\xi+v\eta)}d\xi d\eta$$

$$U_f(u,V)=\frac{Ae^{j\frac{k}{2d}(u^2+V^2)}}{j\lambda d}\frac{f}{d}\int\int_{-\infty}^{\infty}\left\{t_A(\xi,\eta)P\left[\xi(\frac{f}{d}),\eta\left(\frac{f}{d}\right)\right]\right\}e^{-j\frac{2\pi}{\lambda d}(u\xi+v\eta)}d\xi d\eta$$

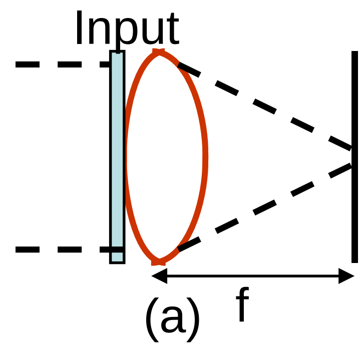


(c)

Again up to a phase factor, the focal-plane amplitude distribution of the input transparency is the Fourier transform of the portion of the input subtended by the projected lens aperture.

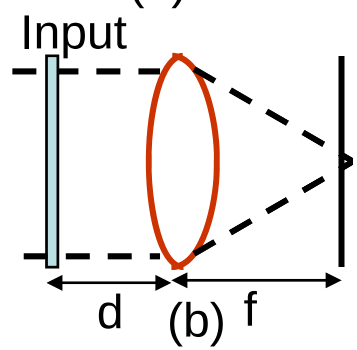
5.2 Comparison of three cases

a) Input in front of the transformer pressed against it:



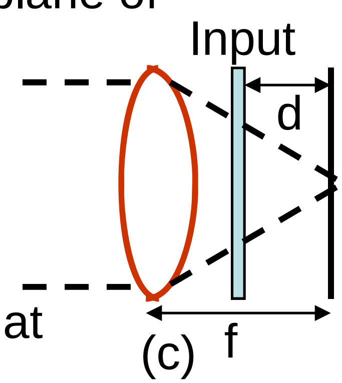
$$U_f(u,V)=\frac{Ae^{j\frac{k}{2f}(u^2+V^2)}}{j\lambda f}\int\int_{-\infty}^{\infty} t_A(x,y)P(x,y)e^{-j\frac{2\pi}{\lambda f}(xu+yV)}dxdy$$

b) Input in front of the transformer at a distance from the lens:



$$\underbrace{U_f(u,V)}_{\text{Amplitude and phase of the light at } (u,V)} = \underbrace{\frac{Ae^{j\frac{k}{2f}\left(1-\frac{d}{f}\right)(u^2+V^2)}}{j\lambda f}}_{\text{Quadratic phase factor Can be eleiminated by } d=f}\underbrace{\int\int_{-\infty}^{\infty} t_A(\xi,\eta)P(x,y)e^{-j\frac{2\pi}{\lambda f}(\xi u+\eta V)}d\xi d\eta}_{\text{Amplitude and phase of the input spectrum at frequencies } f_X=u/\lambda f \text{ and } f_Y=v/\lambda f}$$

c) Input in back of the transformer at a distance d from the back focal plane of the transformer:



$$U_f(u,V)=\frac{Ae^{j\frac{k}{2d}(u^2+V^2)}}{j\lambda d}\frac{f}{d}\int\int_{-\infty}^{\infty}\left\{t_A(\xi,\eta)P\left[\xi(\frac{f}{d}),\eta\left(\frac{f}{d}\right)\right]\right\}e^{-j\frac{2\pi}{\lambda d}(u\xi+v\eta)}d\xi d\eta$$

The results of case (a), (b) and (c) are essentially the same except that in case (c) scale of the Fourier transform is determined by the experimenter.

That is d, the distance between input and back focal plane of the lens.

For $d=f$, both (a) and (c) cases give identical results.

5.3 Image Formation: Monochromatic Illumination

If an object in front of a lens illuminated properly, there will be a second plane across which the field distribution resembles the object.

The image may be

real : actual rays intersect to form the image

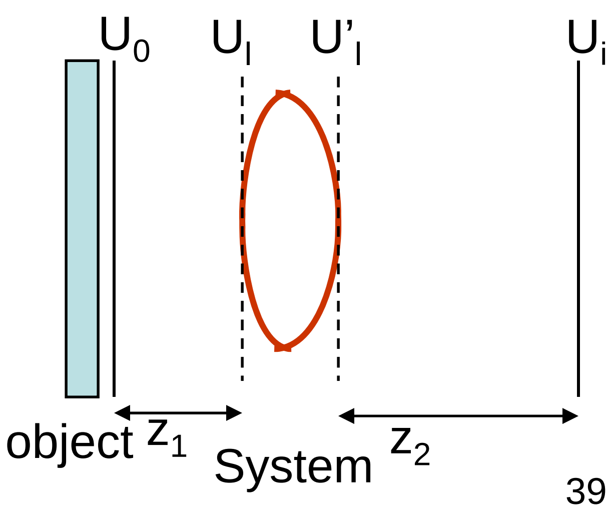
virtual : seems like the rays coming from a virtual intensity distribution plane.

We impose the following limitations on our analysis at this point:

Lenses are positive, aberration-free, and thin that form real images $z_1 > f$.

Only monochromatic illumination (implies that the imaging system is linear in complex amplitutedes).

More general case will be trated in chapter 6.



5.3.1 The impulse Response of a Positive Lens I

$U_0(\xi,\eta)$ complex field imediately behind the object (input transparency) at a distance z_1

$U_i(u,V)$ complex field at a distance z_2 behind the lens

Goal: find the conditions under which the field distribution U_i can reasonably be said to be an *image* of the *object* distribution U_0.

Using the concept of linearity (Chapter 2) we can express the U_i by:

$$U_i(u,V) = \iint_{-\infty}^{\infty} h(u,V;\xi,\eta) U_0(\xi,\eta)\, d\xi d\eta$$

$h(u,V;\xi,\eta)$ is the *impulse response* or the field amplitude producedt at (u,V) by a unit amplitude point source applied at object coordinates.

For an acceptable image $U_i(u,V)$ must be as similar as possible to $U_0(\xi,\eta)$.

Or the impulse response should resemble a Dirac Delta function:

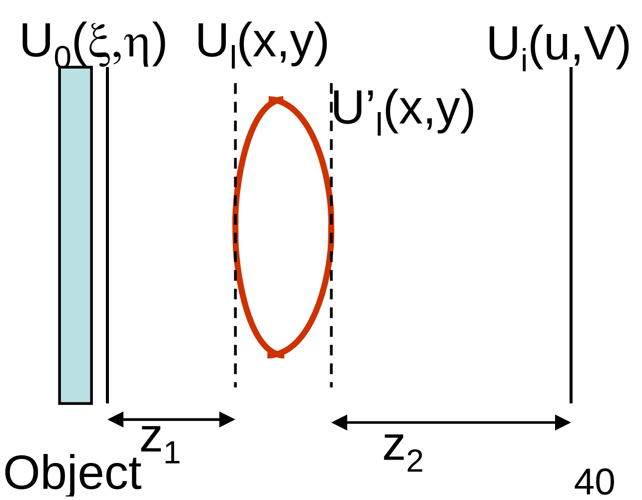
$$h(u,V;\xi,\eta) \approx K\delta(u \pm M\xi, V \pm M\eta)$$

K is a complex constant.

M represents the system magnification

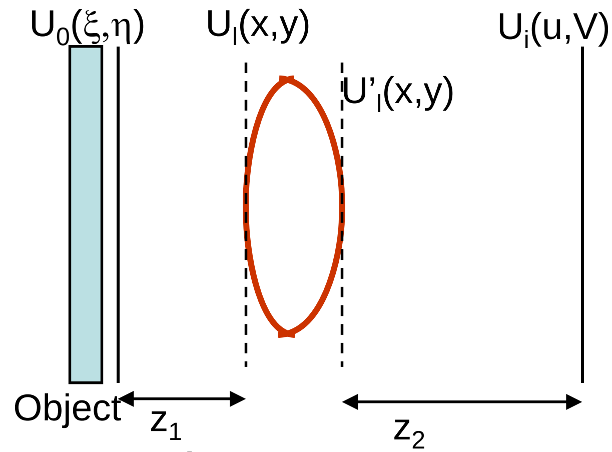
$\pm$ signs accomodate image inversion

Image plane: a plane over which the approximation is the closest or idealy $h(u,V;\xi,\eta) \to K\delta(u \pm M\xi, V \pm M\eta)$.



5.3.1 The impulse Response of a Positive Lens II

To find impulse response: $h(u,V;\xi,\eta)$ we let the object be a unit amplitude point source located at (ξ,η) or $U_0(\xi,\eta)=\dfrac{e^{jkr}}{r}$ .The incident wave on the lens will be a spherical wave diverging from the point (ξ,η).



Paraxial approximation to $\dfrac{e^{jkr}}{r}$ at $r=z_1$ is:

$$U_l(x,y)\approx\frac{1}{j\lambda z_1}e^{j\frac{k}{2z_1}\left[(x-\xi)^2+(y-\eta)^2\right]}$$

The field distribution after the lens: $U_l^{'}(x,y)=U_l(x,y)P(x,y)e^{-j\frac{k}{2f}\left[x^2+y^2\right]}$

Using Fresnel Diffraction equation to account for propagtion over a distance z_2 :

$$h(u,V;\xi,\eta)=\frac{1}{j\lambda z_2}\iint_{-\infty}^{\infty}U_l^{'}(x,y)e^{j\frac{k}{2z_2}\left[(u-x)^2+(V-y)^2\right]}dxdy$$

Putting together all:

$$h(u,V;\xi,\eta)=\frac{1}{\lambda^2 z_1 z_2}\iint_{-\infty}^{\infty}e^{j\frac{k}{2z_1}\left[(x-\xi)^2+(y-\eta)^2\right]}P(x,y)e^{-j\frac{k}{2f}\left[x^2+y^2\right]}e^{j\frac{k}{2z_2}\left[(u-x)^2+(V-y)^2\right]}dxdy$$

5.3.1 The impulse Response of a Positive Lens III

Integration is over x and y so we can pull out all of the terms independent of x, y

$$h(u,V;\xi,\eta)=\frac{1}{\lambda^2 z_1 z_2}\iint_{-\infty}^{\infty} e^{j\frac{k}{2z_1}\left[(x-\xi)^2+(y-\eta)^2\right]} P(x,y) e^{-j\frac{k}{2f}\left[x^2+y^2\right]} e^{j\frac{k}{2z_2}\left[(u-x)^2+(V-y)^2\right]} dxdy$$

Neglecting a pure phase factor we get:

$$\boxed{\begin{aligned} h(u,V;\xi,\eta)&=\frac{1}{\lambda^2 z_1 z_2} e^{j\frac{k}{2z_2}\left[u^2+V^2\right]} e^{j\frac{k}{2z_1}\left[\xi^2+\eta^2\right]} \\ &\times\iint_{-\infty}^{\infty} P(x,y) e^{j\frac{k}{2}\left[\left(\frac{1}{z_1}+\frac{1}{z_2}-\frac{1}{f}\right)\left(x^2+y^2\right)\right]} e^{-jk\left[\left(\frac{\xi}{z_1}+\frac{u}{z_2}\right)x+\left(\frac{\eta}{z_1}+\frac{V}{z_2}\right)y\right]} dxdy \end{aligned}}$$

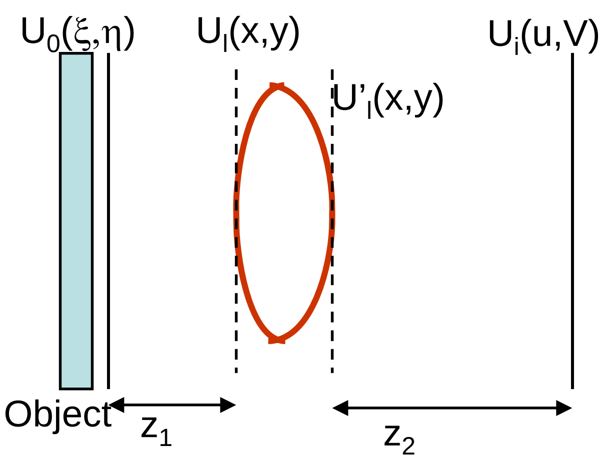
this result together with

$$U_i(u,V)=\iint_{-\infty}^{\infty} h(u,V;\xi,\eta) U_0(\xi,\eta)\, d\xi d\eta$$

enable us to calculate $U_i(u,V)$ if we know $U_0(\xi,\eta)$.

But when we can say $U_i(u,V)$ is image of $U_0(\xi,\eta)$

is difficult unless we do further simplifications.



5.3.2 Eliminating Quadratic Phase Factors: The Lens Law I

Let's look at the troublesome terms of the impulse response mainly quadratic factors:

$$h(u,V;\xi,\eta)=\frac{1}{\lambda^2 z_1 z_2}\underbrace{e^{j\frac{k}{2z_2}\left[u^2+V^2\right]}e^{j\frac{k}{2z_1}\left[\xi^2+\eta^2\right]}}_{\text{independent of the lens coordinates}}$$

$$\times\iint_{-\infty}^{\infty}P(x,y)\underbrace{e^{j\frac{k}{2}\left[\left(\frac{1}{z_1}+\frac{1}{z_2}-\frac{1}{f}\right)\left(x^2+y^2\right)\right]}}_{\text{Depends on the lens coordinates}}\;e^{-jk\left[\left(\frac{\xi}{z_1}+\frac{u}{z_2}\right)x+\left(\frac{\eta}{z_1}+\frac{V}{z_2}\right)y\right]}dxdy$$

Approximations that will elliminate these factors:

1) $e^{j\frac{k}{2}\left[\left(\frac{1}{z_1}+\frac{1}{z_2}-\frac{1}{f}\right)\left(x^2+y^2\right)\right]}$ effect of this term will be broadening of the impulse response. Without that $h(u,V;\xi,\eta)$ will be a pure Fourier transformation of the $P(x,y)$.

So if we choose a distance that the term $e^{j\frac{k}{2}\left[\left(\frac{1}{z_1}+\frac{1}{z_2}-\frac{1}{f}\right)\left(x^2+y^2\right)\right]}$ vanishes then $h(u,V;\xi,\eta)$ would be the closest approximation of the impulse response. So let $\frac{1}{z_1}+\frac{1}{z_2}-\frac{1}{f}=0$

The classical *lens law* of the geometrical optics: $\frac{1}{z_1}+\frac{1}{z_2}-\frac{1}{f}=0$ has to be satisfied for image formation.

5.3.2 Eliminating Quadratic Phase Factors: The Lens Law II

Impulse response after application of the *lens law*:

$$h(u,V;\xi,\eta)=\frac{1}{\lambda^2 z_1 z_2}\underbrace{e^{j\frac{k}{2z_2}\left[u^2+V^2\right]}}_{\text{Depends on the image location}}\underbrace{e^{j\frac{k}{2z_1}\left[\xi^2+\eta^2\right]}}_{\text{Depends only on the object location}}\iint_{-\infty}^{\infty}P(x,y)e^{-jk\left[\left(\frac{\xi}{z_1}+\frac{u}{z_2}\right)x+\left(\frac{\eta}{z_1}+\frac{V}{z_2}\right)y\right]}dxdy$$

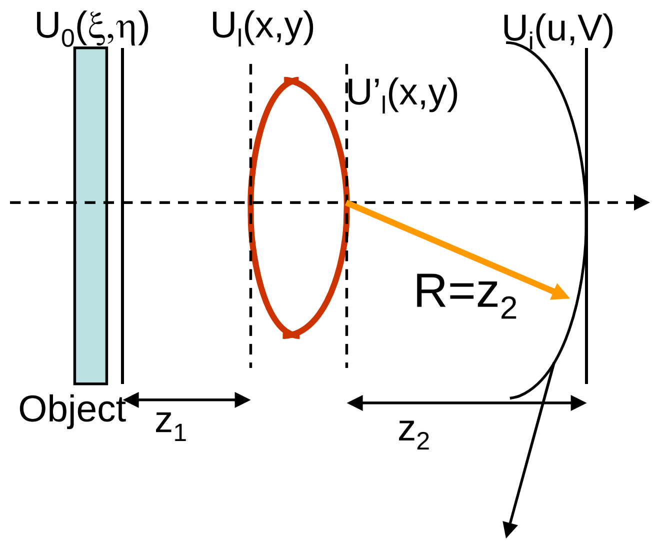
Approximations that will elliminate these factors:

2) $e^{j\frac{k}{2z_2}\left[u^2+V^2\right]}$ can be ignored under two conditions:

a) if we are only interested in the intensity at the object plane, then phase distribution is unimportant. most of the time this is the case

and we drop the $e^{j\frac{k}{2z_2}\left[u^2+V^2\right]}$ factor.

b) The image is measured on a shperical surface with radius z_2, centered at the point where the optical axis pierces the thin lens.



The surface on which the quadratic phase factor on u and V is zero

5.3.2 Eliminating Quadratic Phase Factors III

Impulse response after application of the *lens law* and eliminating the quadratic phase factor dependent on the image coordinates:

$$h(u,V;\xi,\eta)=\frac{1}{\lambda^2 z_1 z_2}\underbrace{e^{j\frac{k}{2z_1}\left[\xi^2+\eta^2\right]}}_{\text{Depends only on the object location}}\iint_{-\infty}^{\infty}P(x,y)e^{-jk\left[\left(\frac{\xi}{z_1}+\frac{u}{z_2}\right)x+\left(\frac{\eta}{z_1}+\frac{V}{z_2}\right)y\right]}dxdy$$

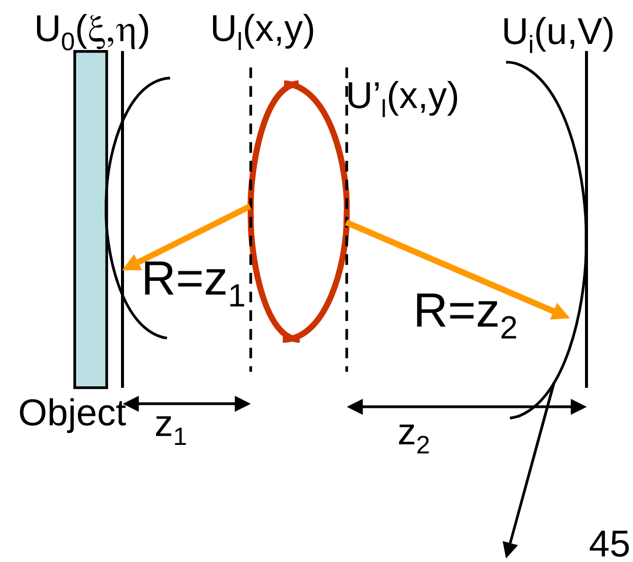
3) $e^{j\frac{k}{2z_1}\left[\xi^2+\eta^2\right]}$ this term will affect the image severly through the convolution operation in calculating the image:

$$U_i\left(u,V\right)=\iint_{-\infty}^{\infty}h\left(u,V;\xi,\eta\right)U_0\left(\xi,\eta\right)d\xi d\eta.$$

Under 3 conditions this term can be ignored:

3.a) the <u>object exists on a shperical surface</u> with radius z_1, centered on the point where the optical axis pierces the thin lens.

<u>This rarely happens in reality.</u>



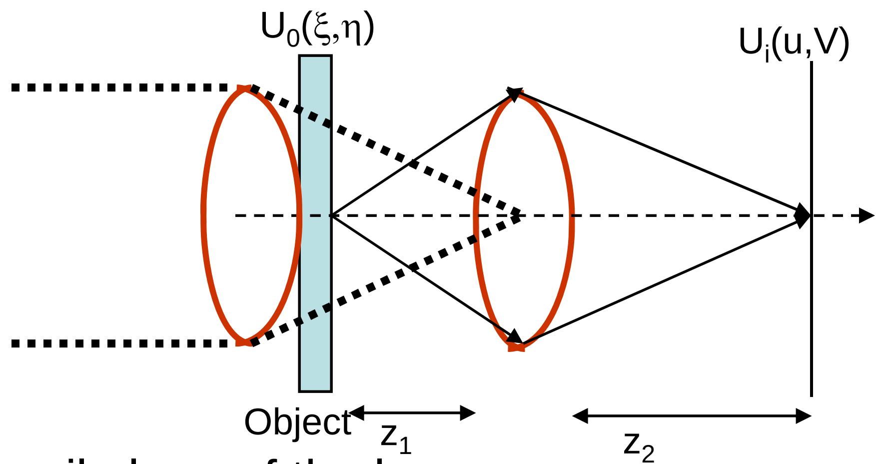
5.3.2 Eliminating Quadratic Phase Factors IV

Impulse response after application of the *lens law* and quadratic phase factor dependent on the image coordinates:

$$h(u,V;\xi,\eta)=\frac{1}{\lambda^2 z_1 z_2}\underbrace{e^{j\frac{k}{2z_1}\left[\xi^2+\eta^2\right]}}_{\text{Depends only on the object location}}\iint_{-\infty}^{\infty}P(x,y)e^{-jk\left[\left(\frac{\xi}{z_1}+\frac{u}{z_2}\right)x+\left(\frac{\eta}{z_1}+\frac{V}{z_2}\right)y\right]}dxdy$$

3.b) the object is illuminated by a spherical wave that is converging towards the point that optical axis pierces the lens.

A proper illumination can make this happen.



The spherical wave illumination results in Fourier transform of the object apear in the pupil plane of the lens.

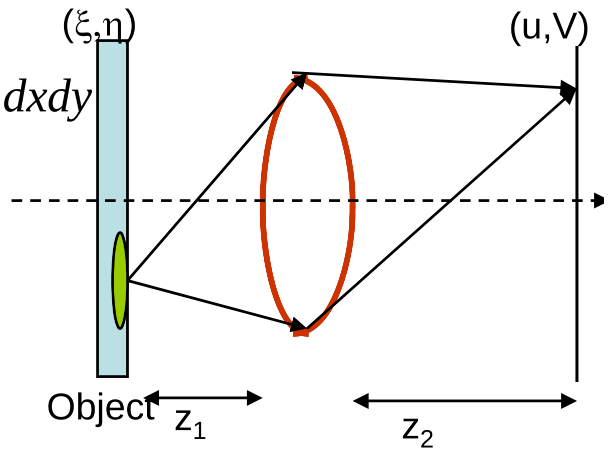
The quadratic phase factor will cancell by the quadratuic phase factor of this converging spherical wave (simmilar to the case of transparency behind the lens:

$$U_f(u,V)=\frac{A\overbrace{e^{j\frac{k}{2d}\left(u^2+V^2\right)}}^{\text{This term with conversion of } u\to\xi \text{ and } V\to\eta \text{ and } d\to -z_1}}{j\lambda d}\frac{f}{d}\int\int_{-\infty}^{\infty}\left\{t_A(\xi,\eta)P\left[\xi(\frac{f}{d}),\eta\left(\frac{f}{d}\right)\right]\right\}e^{-j\frac{2\pi}{\lambda d}(u\xi+v\eta)}d\xi d\eta$$

5.3.2 Eliminating Quadratic Phase Factors V

Impulse response after application of the *lens law* and quadratic phase factor dependent on the image coordinates:

$$h(u,V;\xi,\eta) = \frac{1}{\lambda^2 z_1 z_2} \underbrace{e^{j\frac{k}{2z_1}\left[\xi^2+\eta^2\right]}}_{\text{Depends only on the object location}} \int\int_{-\infty}^{\infty} P(x,y) e^{-jk\left[\left(\frac{\xi}{z_1}+\frac{u}{z_2}\right)x+\left(\frac{\eta}{z_1}+\frac{V}{z_2}\right)y\right]} dxdy$$



3.c) The phase of the quadratic factor changes by an amount that is only a small fraction of a radian within the region of the object that contributes significantly to the field at the particular image point (u,V).

This is true most of the time, otherwise the image at (u,V) would be very blurred.

In that case the $e^{j\frac{k}{2z_1}\left[\xi^2+\eta^2\right]}$ can be replaced by a single phase that depends on the coordinates of the image point of the interest but not the object point.

$$e^{j\frac{k}{2z_1}\left[\xi^2+\eta^2\right]} \rightarrow e^{j\frac{k}{2z_1}\left[\frac{u^2+V^2}{M^2}\right]}$$ where $M = -\frac{z_2}{z_1}$ is the magnification of the system.

Now this new phase factor can be dropped if we are interested only in intesity.

Practical examination shows that this holds for: $\boxed{\text{object size} < \frac{1}{4}(\text{lens aperture size})}$

5.3.2 Eliminating Quadratic Phase Factors

$$U_i(u,V) = \iint_{-\infty}^{\infty} h(u,V;\xi,\eta) U_0(\xi,\eta)\, d\xi\, d\eta$$

$$h(u,V;\xi,\eta) = \frac{1}{\lambda^2 z_1 z_2} \underbrace{e^{j\frac{k}{2z_2}\left[u^2+V^2\right]}}_{\text{Approximation 2}} \underbrace{e^{j\frac{k}{2z_1}\left[\xi^2+\eta^2\right]}}_{\text{Approximation3}} \iint_{-\infty}^{\infty} P(x,y) \underbrace{e^{j\frac{k}{2}\left[\left(\frac{1}{z_1}+\frac{1}{z_2}-\frac{1}{f}\right)\left(x^2+y^2\right)\right]}}_{\substack{\text{Depends on the lens coordinates} \\ \text{Lens law took care of it}}} e^{-jk\left[\left(\frac{\xi}{z_1}+\frac{u}{z_2}\right)x+\left(\frac{\eta}{z_1}+\frac{V}{z_2}\right)y\right]} dxdy$$

Result of all of the approximations on the impulse response of a thin, positive, spherical lens is:

$$h(u,V;\xi,\eta) \approx \frac{1}{\lambda^2 z_1 z_2} \iint_{-\infty}^{\infty} P(x,y) e^{-jk\left[\left(\frac{\xi}{z_1}+\frac{u}{z_2}\right)x+\left(\frac{\eta}{z_1}+\frac{V}{z_2}\right)y\right]} dxdy$$

together with $M = -z_2 / z_1$

$$h(u,V;\xi,\eta) \approx \frac{1}{\lambda z_1} \frac{1}{\lambda z_2} \iint_{-\infty}^{\infty} P(x,y) e^{-j\frac{2\pi}{\lambda z_2}\left[(u-M\xi)x+(V-M\eta)y\right]} dxdy$$

when the lens law is applied we see that the impulse response of a thin lens is the Fraunhofer diffraction pattern of the lens aperture, apart from a scaling factor of $\frac{1}{\lambda z_1}$, centered on the image coordinates of $u = M\xi,\ V = M\eta$. The simplifications used to arrive at this conclusion are:

1) Lens law holds $1/z_1 + 1/z_2 = 1/f$

2) Only intensity of image matters or the image plane is spherical with the radius of curvature equal to the image-lens distance.

3) Object is on a spherical surface or illumination is done by a spherical wave converging with radius of curvature equal to the object-lens distance or object size $< 1/4$ of the lens aperture size.

5.3.3 The Relation Between Object and Image

For a perfect imaging system, image is a magnified/demagnified replica of the object.

In Geometrical optics image and object are related by:

$$U_i\left(u,V\right)=\frac{1}{|M|}U_0\left(\frac{u}{M},\frac{V}{M}\right)$$

We can show that the wave optics image reduces to the geometrical optics image as $\lambda \to 0$. Start from impulse response (approximate form) of a positive lens:

$$\lim_{\lambda\to 0} h(u,V;\xi,\eta) \approx \lim_{\lambda\to 0}\frac{1}{\lambda z_1}\frac{1}{\lambda z_2}\iint_{-\infty}^{\infty} P(x,y)e^{-j\frac{2\pi}{\lambda z_2}\left[(u-M\xi)x+(V-M\eta)y\right]}dxdy$$

Change variables: $x'=x/\lambda z_2,\ y'=y/\lambda z_2$

$$\lim_{\lambda\to 0} h(u,V;\xi,\eta) \approx \lim_{\lambda\to 0}\frac{z_2}{z_1}\iint_{-\infty}^{\infty} P(x'\lambda z_2, y'\lambda z_1)e^{-j2\pi\left[(u-M\xi)x'+(V-M\eta)y'\right]}dx'dy'$$

As $\lambda \to 0$ the aperture function $P(x'\lambda z_2, y'\lambda z_1)$ gets wider.

So at the limit the aperture is infinitly wide. We can replace it with 1. Difining $M = z_2/z_1$

we get: $$\lim_{\lambda\to 0} h(u,V;\xi,\eta) \approx \lim_{\lambda\to 0} M \underbrace{\iint_{-\infty}^{\infty} 1.e^{-j2\pi\left[(u-M\xi)x'+(V-M\eta)y'\right]}dx'dy'}_{\text{Fourier transform of unity}} \approx \frac{1}{M}\delta\left(\xi-\frac{u}{M},\eta-\frac{V}{M}\right)$$

$$U_i\left(u,V\right)=\iint_{-\infty}^{\infty}\lim_{\lambda\to 0} h(u,V;\xi,\eta)U_0\left(\xi,\eta\right)d\xi d\eta$$

$$U_i\left(u,V\right)=\iint_{-\infty}^{\infty}\frac{1}{M}\delta\left(\xi-\frac{u}{M},\eta-\frac{V}{M}\right)U_0\left(\xi,\eta\right)d\xi d\eta \to \boxed{U_i\left(u,V\right)=\frac{1}{M}U_0\left(\frac{u}{M},\frac{V}{M}\right)}$$

Effect of diffraction on the relation between object and image I

Impulse response of the imaging system:

$$h(u,V;\xi,\eta) \approx \frac{1}{\lambda z_1}\frac{1}{\lambda z_2}\iint_{-\infty}^{\infty} P(x,y) e^{-j\frac{2\pi}{\lambda z_2}\left[(u-M\xi)x+(V-M\eta)y\right]} dxdy$$

This is impulse response of a linear space-variant system. So the object and image are related by a superposition integral not a convolution. The space-variant atribute is due to magnification and inversion. To transform this relationship to a convolution:

change to a normalized set of object-plane variables: $\tilde{\xi} = M\xi,\ \tilde{\eta} = M\eta$

$$h(u,V;\tilde{\xi},\tilde{\eta}) = \frac{1}{\lambda^2 z_1 z_2}\iint_{-\infty}^{\infty} P(x,y) e^{-j\frac{2\pi}{\lambda z_2}\left[\left(u-\tilde{\xi}\right)x+\left(V-\tilde{\eta}\right)y\right]} dxdy$$

This relation has a convolution form which depends on only $\left(u-\tilde{\xi},V-\tilde{\eta}\right)$

Another set of normalizing coordinates for further simplification: $\tilde{x} = \frac{x}{\lambda z_2}, \tilde{y} = \frac{y}{\lambda z_2}, \tilde{h} = \frac{1}{|M|}h$

$$h(u,V;\tilde{\xi},\tilde{\eta}) = |M| \iint_{-\infty}^{\infty} P(\lambda z_2\tilde{x},\lambda z_2\tilde{y}) e^{-j2\pi\left[\left(u-\tilde{\xi}\right)\tilde{x}+\left(V-\tilde{\eta}\right)\tilde{y}\right]} d\tilde{x}d\tilde{y} = |M|\,\tilde{h}$$

$$U_i\left(u,V\right) = \iint_{-\infty}^{\infty} \tilde{h}\left(u-\tilde{\xi},V-\tilde{\eta}\right) \underbrace{\left[\frac{1}{|M|}U_0\left(\frac{\tilde{\xi}}{M},\frac{\tilde{\eta}}{M}\right)\right]}_{U_g(u,V)\ \text{image from Geometrical optics analysis}} d\tilde{\xi}d\tilde{\eta}$$

Effect of diffraction on the relation between object and image II

$$U_i(u,V) = \underbrace{\tilde{h}(u,V)}_{\text{Effect of diffraction on the image}} \otimes \underbrace{U_g(u,V)}_{\text{Image by Geometrical optics analysis}}$$ where $U_g(u,V) = \frac{1}{|M|} U_0\left(\frac{\tilde{\xi}}{M}, \frac{\tilde{\eta}}{M}\right)$

The <u>point-spead function introduced by diffraction</u> is then:

$$\tilde{h}(u,V) = \iint_{-\infty}^{\infty} P(\lambda z_2 \tilde{x}, \lambda z_2 \tilde{y}) e^{\left[-j2\pi(u\tilde{x}+V\tilde{y})\right]} d\tilde{x} d\tilde{y}$$ where $\tilde{x} = \frac{x}{\lambda z_2}, \tilde{y} = \frac{y}{\lambda z_2}, \tilde{h} = \frac{1}{|M|} h$

Conclusions:

1) The ideal image produced by a diffraction-limitted optical system (free from aberrations) is scaled and inverted version of the object.

2) The effect of diffraction is <u>to convolve that ideal image with the Fraunhofer diffraction pattern of the lens pupil</u>.

Convolution smooths the image and attenuates the fine details of the object.

We will talk about applications of filtering to imaging systems in chapter 6.

5.4 Analysis of Complex Coherent Optical Systems

- A certain "operator" notation will be useful to analyze complicated optical systems with many lenses and free-space regions.

5.4.1 An operator notation

Several operator approaches has been introduced to analyze complicated optical systems. We follow the approach introduced by Nazarathy and Shamir:

Simplifying assumption:

1) Monochromatic systems.

2) Coherent systems.

3) Only paraxial conditions will be considered.

4) One-dimensional treatment of the system good for the problems that their aperture function is separable in rectangular coordinates.

Oparator: represents a fundamental operation of the system.

Notation: Operator$\left[\text{parameters dependent on the geometry}\right]\left\{\text{Quantity}\right\}$

5.4.1 The basic operators useful to us

Four operators that are sufficient to analyze most optical systems:

1) Multiplication by a quadratic phase exponential: $\mathcal{Q}[c]\{U(x)\} = e^{j\frac{k}{2}cx^2} U(x)$

Where $k = 2\pi/\lambda,$ and c is an inverse length and $\mathcal{Q}[c]^{-1} = \mathcal{Q}[-c]$

2) Scaling by a constant: $\mathcal{V}[b]\{U(x)\} = |b|^{1/2} U(bx)$

Where b is dimensionless and $\mathcal{V}[b]^{-1} = \mathcal{V}[1/b]$

3) Fourier transformation: $\mathcal{F}\{U(x)\} = \int_{-\infty}^{\infty} U(x) e^{-j2\pi fx} dx = G(f)$

$\mathcal{F}^{-1}\{G(f)\} = \int_{-\infty}^{\infty} U(f) e^{j2\pi fx} df$ and $\mathcal{F}\mathcal{F}^{-1}\{U(x)\} = U(x)$

4) Free space propagation: $\mathcal{R}[d]\{U(x_1)\} = \dfrac{1}{\sqrt{j\lambda d}} \int_{-\infty}^{\infty} U(x_1) e^{-j\frac{k}{2d}(x_2 - x_1)^2} dx_1$

Where $d =$ distance of propagation, $x_2 =$ coordinate after propagation.

$\mathcal{R}^{-1}[d] = \mathcal{R}[\text{-}d]$

5.4.1 Relationship among the basic operators

$\mathcal{V}[t_2]\mathcal{V}[t_1] = \mathcal{V}[t_2 t_1]$

$\mathcal{F}\mathcal{V}[t] = \mathcal{V}\left[\frac{1}{t}\right]\mathcal{F}$ a statement of the similarity theorem of the Fourier analysis

$\mathcal{F}\mathcal{F} = \mathcal{V}[-1]$ follows from the Fourier inversion theorem

$\mathcal{Q}[c_2]\mathcal{Q}[c_1] = \mathcal{Q}[c_2 + c_1]$

$\mathcal{R}[d] = \mathcal{F}^{-1}\mathcal{Q}\left[-\lambda^2 d\right]\mathcal{F}$ Free propagation in space can be analyzed by Fresnel diffraction or by a series of Fourier transformation, multiplication by a quadratic phase factor, and inverse Fourier transformation.

$\mathcal{Q}[c]\mathcal{V}[t] = \mathcal{V}[t]\mathcal{Q}\left[\frac{c}{t^2}\right]$ can be shown by writing the definitions

$\mathcal{R}[d] = \mathcal{Q}\left[\frac{1}{d}\right]\mathcal{V}\left[\frac{1}{\lambda d}\right]\mathcal{F}\mathcal{Q}\left[\frac{1}{d}\right]$ Fresnel diffraction operation is equivalent to multiplication by a quadratic phase factor, properly scaled Fourier transform, followed by multiplication by a quadratic-phase exponential.

$\mathcal{V}\left[\frac{1}{\lambda f}\right]\mathcal{F} = \mathcal{R}[f]\mathcal{Q}\left[-\frac{1}{f}\right]\mathcal{R}[f]$

Fields across the front and back focal planes of a positive lens are related by a properly scaled Fourier transform with no quadratic phase exponential.

Relations between the basic operators

	$\mathcal{V}$	$\mathcal{F}$	$\mathcal{Q}$	$\mathcal{R}$
$\mathcal{V}$	$\mathcal{V}[t_2]\mathcal{V}[t_1]=\mathcal{V}[t_2 t_1]$	$\mathcal{V}[t]\mathcal{F}=\mathcal{F}\mathcal{V}\left[\frac{1}{t}\right]$	$\mathcal{V}[t]\mathcal{Q}[c]=\mathcal{Q}\left[t^2 c\right]\mathcal{V}[t]$	$\mathcal{V}[t]\mathcal{R}[d]=\mathcal{R}\left[\frac{d}{t^2}\right]\mathcal{V}[t]$
$\mathcal{F}$	$\mathcal{F}\mathcal{V}[t]=\mathcal{V}\left[\frac{1}{t}\right]\mathcal{F}$	$\mathcal{F}\mathcal{F}=\mathcal{V}[-1]$	$\mathcal{Q}[c]\mathcal{F}=\mathcal{F}\mathcal{R}\left[-\frac{c}{\lambda^2}\right]$	$\mathcal{F}\mathcal{R}[d]=\mathcal{Q}\left[-\lambda^2 d\right]\mathcal{F}$
$\mathcal{Q}$	$\mathcal{Q}[c]\mathcal{V}[t]=\mathcal{V}[t]\mathcal{Q}\left[\frac{c}{t^2}\right]$	$\mathcal{Q}[c]\mathcal{F}=\mathcal{F}\mathcal{R}\left[-\frac{c}{\lambda^2}\right]$	$\mathcal{Q}[c_2]\mathcal{Q}[c_1]=\mathcal{Q}[c_2+c_1]$	$\mathcal{Q}[c]\mathcal{R}[d]=\mathcal{R}\left[\left(d^{-1}+c\right)^{-1}\right]$ $\cdot\mathcal{V}[1+cd]\cdot\mathcal{Q}\left[\left(c^{-1}+d\right)^{-1}\right]$
$\mathcal{R}$	$\mathcal{R}[d]\mathcal{V}[t]=\mathcal{V}[t]\mathcal{R}\left[t^2 d\right]$	$\mathcal{R}[d]\mathcal{F}=\mathcal{F}\mathcal{Q}\left[-\lambda^2 d\right]$	$\mathcal{R}[d]\mathcal{Q}[c]=\mathcal{Q}\left[\left(c^{-1}+d\right)^{-1}\right]$ $\cdot\mathcal{V}\left[\left(1+cd\right)^{-1}\right]\cdot\mathcal{R}\left[\left(d^{-1}+c\right)^{-1}\right]$	$\mathcal{R}[d_1]\mathcal{R}[d_2]=\mathcal{R}[d_1+d_2]$

5.4.2 Application of the Operator Approach to Some Optical Systems I

Example 1) Two spherical lenses each with focal length of f and separation of f

<u>Goal:</u> to determione the relationship between the complex field across S_2 and S_1.

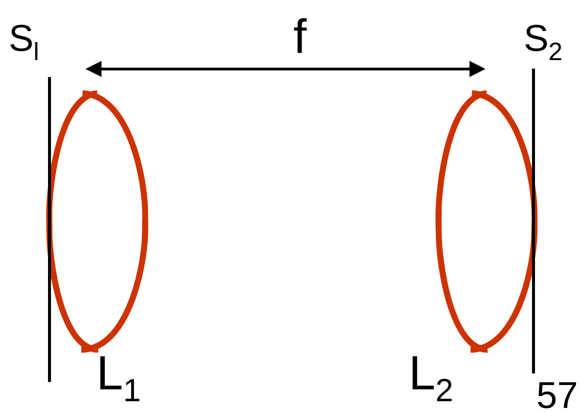
The reletionship operator: $\mathcal{S} = \underbrace{\mathcal{Q}\left[-\frac{1}{f}\right]}_{\text{Lens 2}} \underbrace{\mathcal{R}[f]}_{\text{Propagation}} \underbrace{\mathcal{Q}\left[-\frac{1}{f}\right]}_{\text{Lens 1}}$

We can simplify the $\mathcal{S}$ using $\mathcal{R}[d] = \mathcal{Q}\left[\frac{1}{d}\right]\mathcal{V}\left[\frac{1}{\lambda d}\right]\mathcal{F}\mathcal{Q}\left[\frac{1}{d}\right]$

$\mathcal{S} = \mathcal{Q}\left[-\frac{1}{f}\right]\mathcal{Q}\left[\frac{1}{f}\right]\mathcal{V}\left[\frac{1}{\lambda f}\right]\mathcal{F}\mathcal{Q}\left[\frac{1}{f}\right]\mathcal{Q}\left[-\frac{1}{f}\right]$ using $\mathcal{Q}\left[-\frac{1}{f}\right]\mathcal{Q}\left[\frac{1}{f}\right] = \mathcal{Q}\left[\frac{1}{f}\right]\mathcal{Q}\left[-\frac{1}{f}\right] = 1$

$\mathcal{S} = 1 \cdot \mathcal{V}\left[\frac{1}{\lambda f}\right]\mathcal{F} \cdot 1 \rightarrow \mathcal{S} = \mathcal{V}\left[\frac{1}{\lambda f}\right]\mathcal{F}$ this is equivalent to a scaled Fourier transform without a quadratic phase factor. This is similar to the focal-plane-to-focal-plane relationship seen earlier:

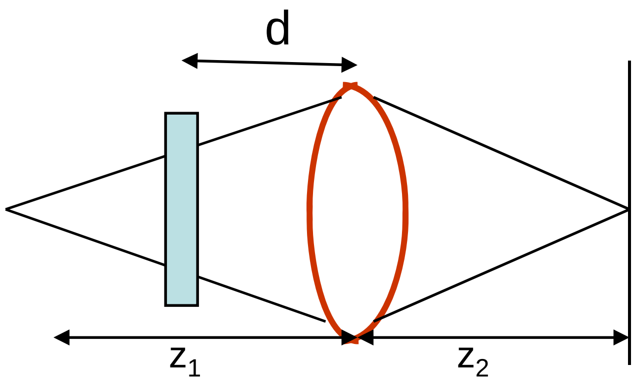
$$\underbrace{U_f(u)}_{\text{Field just to the right of } L_2} = \frac{1}{\sqrt{\lambda f}} \int_{-\infty}^{\infty} \underbrace{U_0(x)}_{\text{Field just to the left of } L_1} e^{-j\frac{k}{f}xu} dx$$



5.4.2 Application of the Operator Approach to Some Optical Systems II

Example 2) An input transparency located at a distance d from a converging lens with focal length of f illuminated by a point source located at a distance z_1 from the lens $(z_1 > d)$. Location of the output of interest z_2: on the plane of image of the point source. Where the usual lens law holds $1/z_1 + 1/z_2 = 1/f$

$$\mathcal{S} = \underbrace{\mathcal{R}[z_2]}_{\text{Propagation}} \underbrace{\mathcal{Q}\left[-\frac{1}{f}\right]}_{\text{Effect of the lens}} \underbrace{\mathcal{R}[d]}_{\text{Propagation}} \underbrace{\mathcal{Q}\left[\frac{1}{z_1 - d}\right]}_{\text{Input is illuminated by a diverging spherical wave that affects it by a quadratic phase factor}}$$



Applying the the lens law:

$$\mathcal{S} = \mathcal{R}[z_2]\mathcal{Q}\left[-\frac{1}{z_1} - \frac{1}{z_2}\right]\mathcal{R}[d]\mathcal{Q}\left[\frac{1}{z_1 - d}\right]$$ Using the table we simplify this

$$\mathcal{R}[d]\mathcal{Q}[c] = \mathcal{Q}\left[\left(c^{-1} + d\right)^{-1}\right] \cdot \mathcal{V}\left[(1 + cd)^{-1}\right] \cdot \mathcal{R}\left[\left(d^{-1} + c\right)^{-1}\right]$$

$$\mathcal{R}[z_2]\mathcal{Q}\left[-\frac{1}{z_1} - \frac{1}{z_2}\right] = \mathcal{Q}\left[\left(-\frac{z_1 z_2}{z_1 + z_2} + z_2\right)^{-1}\right]\mathcal{V}\left[\left(1 - \frac{z_2}{z_1} - 1\right)^{-1}\right]\mathcal{R}\left[\left(\frac{1}{z_2} - \frac{1}{z_1} - \frac{1}{z_2}\right)^{-1}\right]$$

$$\mathcal{R}[z_2]\mathcal{Q}\left[-\frac{1}{z_1} - \frac{1}{z_2}\right] = \mathcal{Q}\left[\left(\frac{z_1 + z_2}{z_2^2}\right)\right]\mathcal{V}\left[-\frac{z_1}{z_2}\right]\mathcal{R}[-z_1]$$

5.4.2 Application of the Operator Approach to Some Optical Systems III

$$\mathcal{S}=\mathcal{Q}\left[\left(\frac{z_1+z_2}{z_2^2}\right)\right]\mathcal{V}\left[-\frac{z_1}{z_2}\right]\mathcal{R}\left[-z_1\right]\mathcal{R}\left[d\right]\mathcal{Q}\left[\frac{1}{z_1-d}\right]=\mathcal{Q}\left[\left(\frac{z_1+z_2}{z_2^2}\right)\right]\mathcal{V}\left[-\frac{z_1}{z_2}\right]\mathcal{R}\left[d-z_1\right]\mathcal{Q}\left[\frac{1}{z_1-d}\right]$$

Next we use $\mathcal{R}\left[d\right]=\mathcal{Q}\left[\frac{1}{d}\right]\mathcal{V}\left[\frac{1}{\lambda d}\right]\mathcal{F}\mathcal{Q}\left[\frac{1}{d}\right]$ to simplify $\mathcal{R}\left[d-z_1\right]$

$$\mathcal{R}\left[d-z_1\right]=\mathcal{Q}\left[\frac{1}{d-z_1}\right]\mathcal{V}\left[\frac{1}{\lambda\left(d-z_1\right)}\right]\mathcal{F}\mathcal{Q}\left[\frac{1}{d-z_1}\right]$$

$$\mathcal{S}=\mathcal{Q}\left[\frac{z_1+z_2}{z_2^2}\right]\mathcal{V}\left[-\frac{z_1}{z_2}\right]\mathcal{Q}\left[\frac{1}{d-z_1}\right]\mathcal{V}\left[\frac{1}{\lambda\left(d-z_1\right)}\right]\mathcal{F}\underbrace{\mathcal{Q}\left[\frac{1}{d-z_1}\right]\mathcal{Q}\left[\frac{1}{z_1-d}\right]}_{1}$$

$$\mathcal{S}=\mathcal{Q}\left[\frac{z_1+z_2}{z_2^2}\right]\mathcal{V}\left[-\frac{z_1}{z_2}\right]\mathcal{Q}\left[\frac{1}{d-z_1}\right]\mathcal{V}\left[\frac{1}{\lambda\left(d-z_1\right)}\right]\mathcal{F}$$

Finaly using $\mathcal{Q}\left[c\right]\mathcal{V}\left[t\right]=\mathcal{V}\left[t\right]\mathcal{Q}\left[\frac{c}{t^2}\right]$ we flip the orders of $\mathcal{Q}$ and $\mathcal{V}$

$$\mathcal{S}=\mathcal{Q}\left[\frac{z_1+z_2}{z_2^2}\right]\mathcal{Q}\left[\frac{z_1^2}{z_2^2\left(d-z_1\right)}\right]\mathcal{V}\left[-\frac{z_1}{z_2}\right]\mathcal{V}\left[\frac{1}{\lambda\left(d-z_1\right)}\right]\mathcal{F}$$

$$\mathcal{S}=\mathcal{Q}\left[\frac{z_1+z_2}{z_2^2}+\frac{z_1^2}{z_2^2\left(d-z_1\right)}\right]\mathcal{V}\left[-\frac{z_1}{z_2}\frac{1}{\lambda\left(d-z_1\right)}\right]\mathcal{F}$$

5.4.2 Application of the Operator Approach to Some Optical Systems IV

$$\mathcal{S} = \mathcal{Q}\left[\frac{d(z_1+z_2)-z_1z_2}{z_2^2(d-z_1)}\right]\mathcal{V}\left[\frac{z_1}{z_2\lambda(z_1-d)}\right]\mathcal{F}$$

Comparing this to the conventional relation ship between $U_1(\xi)$ and $U_2(u)$:

$$U_2(u) = \frac{e^{\left[j\frac{k}{2}\frac{(z_1+z_2)d-z_1z_2}{z_2^2(d-z_1)}\right]}}{\sqrt{\frac{\lambda z_2(z_1-d)}{z_1}}} \underbrace{\int_{-\infty}^{\infty} U_1(\xi) e^{\left[-j\frac{2\pi}{\lambda}\frac{z_1}{z_2(z_1-d)}u\xi\right]} d\xi}_{\text{Fourier transform of the input amplitude distribution with a scaling factor}}$$

Some results from this example that are more general than just this example:

a) Fourier transform plane need not to be the focal plane of the lens that performs the transformation.

b) Fourier trnsform always appears in the plane where the source is imaged.

c) Quadratic-phase factor preceding the Fourier transform operation is always the quadratic-phase factor that would result <u>at the transform plane from a point source of light</u> located on the optical axis in the plane of input transparency.

Operator technique allows a more mathematical analysis of the complicated systems but it is more abstract than the diffraction integrals and furthere away from the physical analysis.