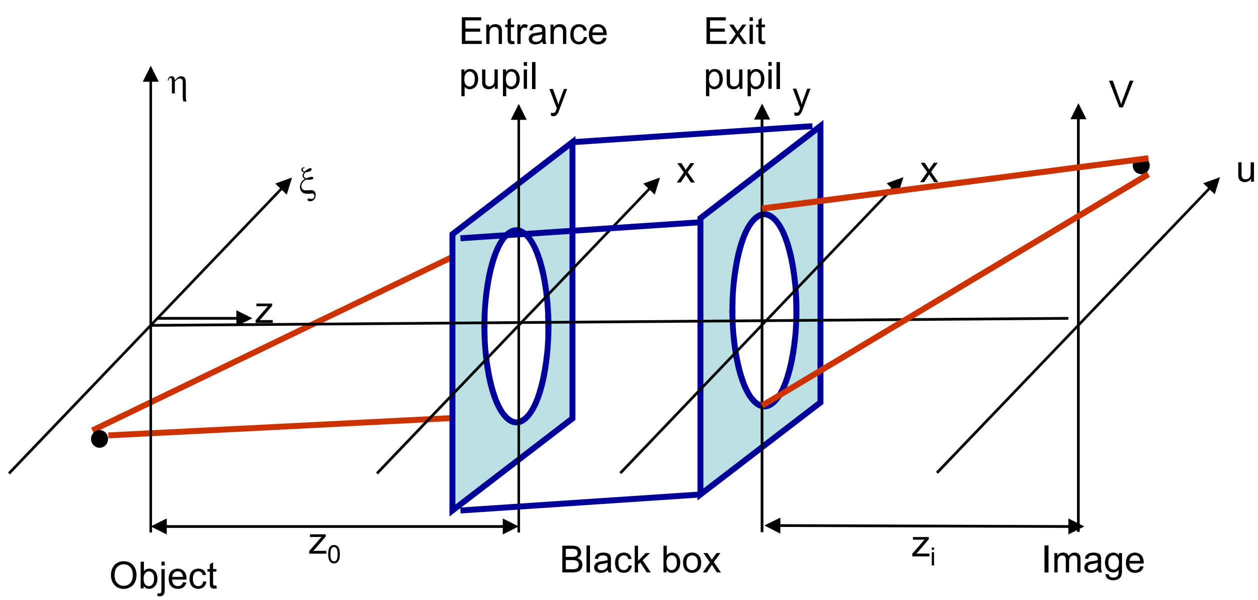
Frequency Analysis of Optical Imaging Systems

Frequency Analysis of Optical Imaging Systems

- Frequency analysis and linear systems theory are relatively new to optics but they have a very fundamental place in the theory of imaging systems
- Introduction of Fourier analysis to optical systems
  - Earnest Abbe (1840-1905) and Loard Rayleigh (1842-1919) Laid the foundations of the Fourier optics
  - P. M.Duffieux in France in1930s published a book on Fourier optics in 1946 translated to English “The Fourier transform and its applications to optics” Wiley 1983
  - Otto Schade in US in 1948 employed methods of linear system theory in analysis and improvement of TV camera lenses.
  - H.H. Hopkins in UK used transfer function methods for the assessment of the quality of optical imaging systems
- This chapter:
  - Coherent imaging systems important in microscopy, holography
  - Incoherent imaging systems wider applications everything else

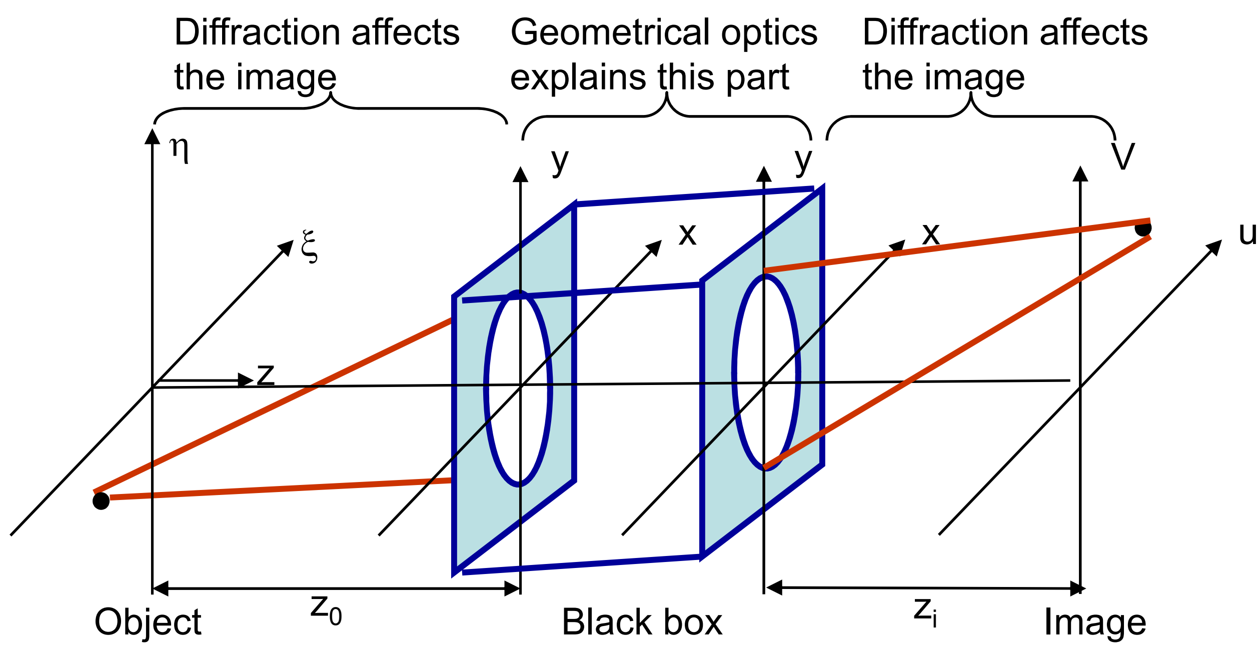
6.1 Generalized treatment of optical imaging systems

- Treatment of more general lens systems
- Treatment of
  - Quasi-monochromatic systems
    - Spatially coherent
    - Spatially incoherent



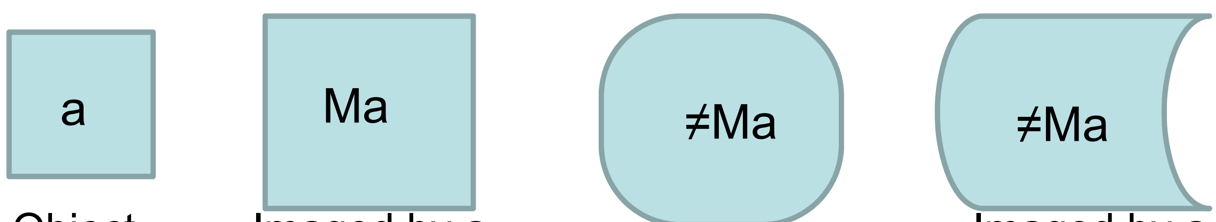
6.1.1 A generalized model

- System: some positive and negative lenses with various distances between them possibly thick.
- Assumption: the system ultimately produces a real image in space.
- If a system is producing virtual images, it can be converted to reals image by lens.
- All imaging elements are lumped into a single "black box" (aggregate system).
- The aggregate system is defined by its "terminal properties" at the planes containing the entrance and exit pupils.
- Sources of the diffraction effects within the system: entrance and exit pupils (conjugate planes) that are images of the physically limiting apertures.
- We assume the passage of light between the entrance and exit pupils is properly described by geometrical optics



Diffraction limited imaging systems

- The following is true for a diffraction-limited imaging system:
  - Spherical wave of a point-source is converted by the system to a perfectly spherical wave that converges towards the image point on a image plane.
  - Location of the image points on the entire image plane is related to the location of the object points on the object plane by a single scaling factor.
  - Location of the image plane stays the same for all points on the image field of interest.
  - Terminal property of a diffraction limited system: converting a diverging spherical wave at the entrance pupil to a converging spherical wave at the exit pupil.
  - A system can be diffraction limited only over a limited region of the object plane.
- Aberration: if the wavefront for a point source on the image plane at the exit pupil departs significantly from an ideal spherical wave, then the system is said to have aberration.
- Aberrations lead to defects in the spatial-frequency response of the imaging system.

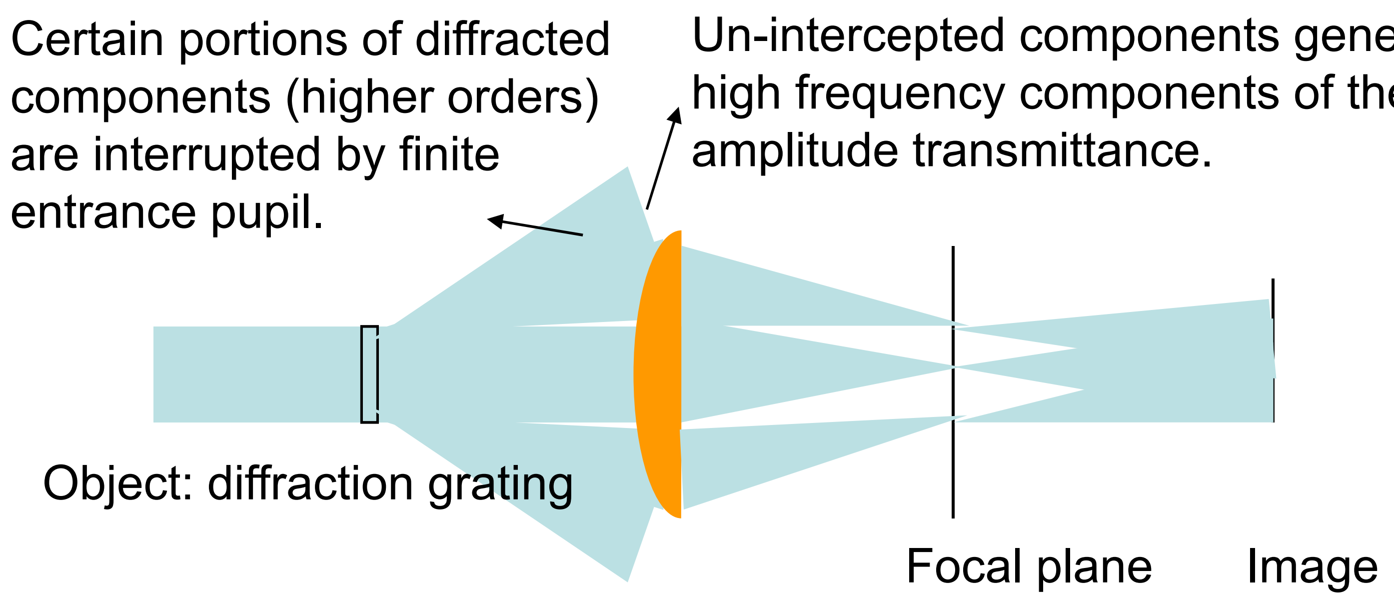


Object | Imaged by a diffraction limited system | Imaged by a system with aberration | Imaged by a system with aberration

6.1.2 Effects of diffraction on the image I

- For determining the source of diffraction in imaging systems two view points are common:
  - The finite entrance pupil seen from the object plane (Abbe 1873)
  - The finite exit pupil seen from the image plane (Rayleigh 1893)
- Because the two pupils are conjugate of each other (image of each other) the two view points are equivalent.

Abbe's view point



6.1.2 Effects of diffraction on the image II Rayleigh's viewpoint

Image amplitude superposition integral:

$$U_i(u,V) = \iint_{-\infty}^{\infty} \underbrace{h(u,V;\xi,\eta)}_{\substack{\text{Amplitude at image coordinates } (u,V) \\ \text{in response to a point-source at } (\xi,\eta)}} \underbrace{U_0(\xi,\eta)}_{\substack{\text{Amplitude distribution} \\ \text{transmitted by the object}}} d\xi d\eta$$

When there is no aberration h arises from a spherical wave converging from the exit pupil towards the ideal image point at $u = M\xi$ and $V = M\eta$

The light amplitude about the ideal image point is the Fraunhofer diffraction pattern of the exit pupil, centered on image coordinates $u = M\xi$ and $V = M\eta$

$$h(u,V;\xi,\eta) = \frac{A}{\lambda z_i} \iint_{-\infty}^{\infty} P(x,y) e^{-j\frac{2\pi}{\lambda z_i}(u-M\xi)x+(V-M\eta)y} dxdy$$

$$P(x,y) = \begin{cases} 1 & \text{inside the aperture} \\ 0 & \text{outside the aperture} \end{cases}$$ is the pupil function and

z_i is the distance from the exit pupil to the image plane

x, y are the coordinates in the plane of exit pupil

The quadratic phase factors in the object and image planes are ignored.

6.1.2 Effects of diffraction on the image III

To achieve space-invariance in the imaging operation, we need to remove effects of magnification and inversion from the equation for h by the following transformation:

$$\tilde{\xi} = M\xi, \quad \tilde{\eta} = M\eta$$

$$h\left(u-\tilde{\xi}, V-\tilde{\eta}\right) = \frac{A}{\lambda z_i}\iint_{-\infty}^{\infty} P(x,y)\, e^{-j\frac{2\pi}{\lambda z_i}\left[\left(u-\tilde{\xi}\right)x+\left(V-\tilde{\eta}\right)y\right]}dxdy$$

Ideal image: the geometrical optics prediction of the image for a perfect system

$$U_g\left(\tilde{\xi},\tilde{\eta}\right) = \frac{1}{|M|}U_0\left(\frac{\tilde{\xi}}{M},\frac{\tilde{\eta}}{M}\right)$$

$$U_i(u,V) = \iint_{-\infty}^{\infty} h\left(u-\tilde{\xi}, V-\tilde{\eta}\right)U_g\left(\tilde{\xi},\tilde{\eta}\right)d\tilde{\xi}d\tilde{\eta}$$

$$\boxed{\underbrace{U_i(u,V)}_{\text{Image by a diffraction-limited system}} = \underbrace{h(u,V)}_{\text{impluse response or Fraunhofer diffraction pattern of the exit pupil}} \otimes \underbrace{U_g\left(\tilde{\xi},\tilde{\eta}\right)}_{\text{image predicted by geometrical optics}}}$$

Where $h(u,V) = \frac{A}{\lambda z_i}\iint_{-\infty}^{\infty} P(x,y)\, e^{-j\frac{2\pi}{\lambda z_i}(ux+Vy)}dxdy$

6.2.3 Polychromic illumination: the coherent and incoherent cases (heuristic approach)

There is no perfectly monochromatic source in nature or in the lab.

The time variations of illumination amplitude and phase have statistical nature.

These fluctuations influence behavior of the imaging systems.

Monochromatic case: the amplitude and phase of the field presented by a complex phasor that is function of the space coordinates (no time dependence).

Polychromatic narrowband case: the amplitude and phase of the field presented by a complex time-varying phasor that is function of the space coordinates and time.

Only statistical techniques can produce a satisfactory explanation of the image produced by such sources.

1) Spatially coherent sources: the phasor amplitudes of the field at all object points and thus the various impulse responses on the image plane vary in unison or correlated fashion.

A coherent imaging system is linear in complex amplitude $\left(U = \sum U_i \rightarrow I = \left(\sum U_i\right)^2\right)$ and the monochromatic analysis holds. The U is a time-invariant phasor that depends on the relative phases of the light. A coherent source can be obtained from lasers and some arc lamps.

2) Spatially incoherent sources: the phasor amplitudes of the field at all object points vary in totally uncorrelated fashin. Obtained from diffused or extended sources.

An incoherent source is linear in intensity or power $\left(I = \sum I_i = \sum U_i^{\,2}\right)$.

6.2.3 Polychromic illumination: the coherent and incoherent cases (rigorous approach)

The phasor representation is obtained by suppressing the positive frequency components of the cosinosoidal field and doubling the negative frequency components.

Polychromatic wave represented by only negative-frequency components:

$u_{-}(P,t) = U(P,t)e^{-j2\pi\bar{\nu}t}$ where $U(P,t)$ is the time-varying phasor representation of $u(P,t)$

Where $\bar{\nu}$ is the center frequency.

For a narrowband system, $(\Delta\nu << \bar{\nu})$, the amplitude impulse response dose not change dramatically for the various frequencies i.e. we can ignore the wavelength-dependence of the spread of the Fraunhofer diff. pattern of a point source which is ok for narrow-band signals. Now we can express the time-varying phasors representing the image as convolution of a wavelength-independent impulse response with the time-varying phasor representation of the object.

$$\underbrace{U_i(u,V;t)}_{\substack{\text{Time varying phasor}\\ \text{representation of the image}}} = \iint_{-\infty}^{\infty} \underbrace{h(u-\tilde{\xi},V-\tilde{\eta})}_{\substack{\text{Wavelength-independent}\\ \text{impulse response}}} \underbrace{U_g(\tilde{\xi},\tilde{\eta};t-\tau)}_{\substack{\text{Time varying phasor representation}\\ \text{of the object in reduced coordinates}}} d\tilde{\xi}d\tilde{\eta}$$

Where $h(u-\tilde{\xi},V-\tilde{\eta}) = \dfrac{A}{\lambda z_i}\iint_{-\infty}^{\infty} P(x,y)e^{-j\frac{2\pi}{\lambda z_i}\left[(u-\tilde{\xi})x+(V-\tilde{\eta})y\right]}dxdy$ and

$U_g(\tilde{\xi},\tilde{\eta}) = \dfrac{1}{|M|}U_0\left(\dfrac{\tilde{\xi}}{M},\dfrac{\tilde{\eta}}{M}\right)$ and τ is the time delay for propagation from $(\tilde{\xi},\tilde{\eta})$ to (u,V)

6.2.3 Polychromic illumination: the coherent and incoherent cases (rigorous approach)

To calculate the image intensity we must time average the instantaneous intensity represented by $\left|U_i\left(u,V;t\right)\right|^2$, due to the fact that the detector integration time $(\sim 50\ ps)$ is extremely long compared to the reciprocal of the bandwidth even for narrowband optical sources $(\Delta\lambda = 1\ nm,\ 1/\Delta f = \lambda^2/c\Delta\lambda \simeq 100\ fs)$.

Therefore $I_i\left(u,V\right) = \left\langle \left|U_i\left(u,V;t\right)\right|^2 \right\rangle_t$ the time average of instantaneous intensity.

$$I_i\left(u,V\right) = \iint_{-\infty}^{\infty} d\tilde{\xi}_1 d\tilde{\eta}_1 \iint_{-\infty}^{\infty} d\tilde{\xi}_2 d\tilde{\eta}_2 h\left(u-\tilde{\xi}_1, V-\tilde{\eta}_1\right) h^*\left(u-\tilde{\xi}_2, V-\tilde{\eta}_2\right)$$
$$\times \left\langle U_g\left(\tilde{\xi}_1,\tilde{\eta}_1;t-\tau_1\right) U_g^*\left(\tilde{\xi}_2,\tilde{\eta}_2;t-\tau_2\right)\right\rangle_t$$

For a fixed image point, h is non-zero only for small region about the ideal image point. So $\left(\tilde{\xi}_1,\tilde{\eta}_1\right)$ and $\left(\tilde{\xi}_2,\tilde{\eta}_2\right)$ are very close to each other and the difference between the time delays τ_1 and τ_2 is negligible under the naorrowband condition.

$$I_i\left(u,V\right) = \iint_{-\infty}^{\infty} d\tilde{\xi}_1 d\tilde{\eta}_1 \iint_{-\infty}^{\infty} d\tilde{\xi}_2 d\tilde{\eta}_2 h\left(u-\tilde{\xi}_1, V-\tilde{\eta}_1\right) h^*\left(u-\tilde{\xi}_2, V-\tilde{\eta}_2\right)$$
$$\times J_g\left(\tilde{\xi}_1,\tilde{\eta}_1;\tilde{\xi}_2,\tilde{\eta}_2\right)$$

Where $J_g\left(\tilde{\xi}_1,\tilde{\eta}_1;\tilde{\xi}_2,\tilde{\eta}_2\right) = \left\langle U_g\left(\tilde{\xi}_1,\tilde{\eta}_1;t\right) U_g^*\left(\tilde{\xi}_2,\tilde{\eta}_2;t\right)\right\rangle_t$ is the "mutual intensity" and it is a measure of the spatial coherence of the light at the two object points.

6.2.3 Polychromic illumination: the coherent and incoherent cases (rigorous approach)

For a perfectly coherent illumination, the time-varying phasor amplitudes across the object plane vary only by a complex constant. So we can write:

$$U_g\left(\tilde{\xi}_1,\tilde{\eta}_1;t\right)=\underbrace{U_g\left(\tilde{\xi}_1,\tilde{\eta}_1\right)}_{\text{Complex Constant}}\frac{\overbrace{U_g\left(0,0;t\right)}^{\text{Time varying phasor amplitude at the origin}}}{\left\langle\left|U_g\left(0,0;t\right)\right|^2\right\rangle^{1/2}}\text{ and }U_g\left(\tilde{\xi}_2,\tilde{\eta}_2;t\right)=\underbrace{U_g\left(\tilde{\xi}_2,\tilde{\eta}_2\right)}_{\text{Complex Constant}}\frac{U_g\left(0,0;t\right)}{\left\langle\left|U_g\left(0,0;t\right)\right|^2\right\rangle^{1/2}}$$

$$J_g\left(\tilde{\xi}_1,\tilde{\eta}_1;\tilde{\xi}_2,\tilde{\eta}_2\right)=U_g\left(\tilde{\xi}_1,\tilde{\eta}_1\right)U_g^*\left(\tilde{\xi}_2,\tilde{\eta}_2\right)\times 1 \text{ the new "mutual intensity"}$$

$$\boxed{I_i\left(u,V\right)=\left|\iint_{-\infty}^{\infty}h\left(u-\tilde{\xi},V-\tilde{\eta}\right)U_g(\tilde{\xi},\tilde{\eta})d\tilde{\xi}d\tilde{\eta}\right|^2=\left|\underbrace{U_i\left(u,V\right)}_{\text{Amplitude convolution equation}}\right|^2}\leftarrow\text{Coherent case}$$

When the object illumination is perfectly incoherent, the phasor amplitudes acros the object vary in statistically independent fashion. This property is represented by:

$$\left\langle U_g(\tilde{\xi}_1,\tilde{\eta}_1;t)U_g^*(\tilde{\xi}_2,\tilde{\eta}_2;t)\right\rangle=\underbrace{\kappa}_{\text{Real constnt}}\underbrace{I_g(\tilde{\xi}_1,\tilde{\eta}_1)}_{\text{Intensity at point 1}}\underbrace{\delta\left(\tilde{\xi}_1-\tilde{\xi}_2,\tilde{\eta}_1-\tilde{\eta}_2\right)}_{\text{delta function vanishes as points 1 and 2 separate}}$$

For incoherent illumination the image intensity is found as a convolution of the intensity impulse response $|h|^2$ with the ideal image intensity I_g

$$\boxed{I_i\left(u,V\right)=\kappa\iint_{-\infty}^{\infty}\left|h\left(u-\tilde{\xi},V-\tilde{\eta}\right)\right|^2 I_g(\tilde{\xi},\tilde{\eta})d\tilde{\xi}d\tilde{\eta}}$$

6.2 Frequency response for diffraction-limited coherent imaging

A coherent imaging system is linear in complex amplitude.

Therefore the intensity mapping will be nonlinear.

The frequency analysis should be applied to the linear amplitude mapping

6.2.1 The amplitude transfer function

Analysis of the coherent systems has yielded a space-invariant form of amplitude mapping. The amplitude at the image plane are convolution of the geometrical image amplitudes with a space-invariant impulse response.

$$U_i\left(u,V\right) = \iint_{-\infty}^{\infty} h\left(u-\tilde{\xi}, V-\tilde{\eta}\right) U_g\left(\tilde{\xi},\tilde{\eta}\right) d\tilde{\xi} d\tilde{\eta}$$

$$\mathcal{F}\left\{U_i\left(u,V\right)\right\} = \mathcal{F}\left\{h\left(u,V\right)\right\} \mathcal{F}\left\{U_g\left(u,V\right)\right\}$$

We will apply transfer-function concept to this picture.

Defining the frequency spectra of input and output and aplitude transfer function:

$$G_g\left(f_X, f_Y\right) = \iint_{-\infty}^{\infty} U_g\left(u,V\right) e^{-j2\pi\left(f_X u + f_Y V\right)} dudV$$

$$G_i\left(f_X, f_Y\right) = \iint_{-\infty}^{\infty} U_i\left(u,V\right) e^{-j2\pi\left(f_X u + f_Y V\right)} dudV$$

$$H\left(f_X, f_Y\right) = \iint_{-\infty}^{\infty} h\left(u,V\right) e^{-j2\pi\left(f_X u + f_Y V\right)} dudV$$

Applying the convolution theorem

$$G_i\left(f_X, f_Y\right) = H\left(f_X, f_Y\right) G_g\left(f_X, f_Y\right)$$

This is effect of diffraction-limited imaging in the frequency domaine.

6.2.1 The amplitude transfer function and physical characteristics of the system

$$H\left(f_X, f_Y\right) = \iint_{-\infty}^{\infty} h(u,V)\, e^{-j2\pi\left(f_X u + f_Y V\right)} du dV$$

Note H is defined as the Fourier transform of the amplitude point-spread function.

We can aslo express H as the Fraunhofer diffraction pattern and is Fourier transform of the pupil function (eq 6-5).

$$H\left(f_X, f_Y\right) = \mathcal{F}\left\{ \frac{A}{\lambda z_i} \iint_{-\infty}^{\infty} P(x,y)\, e^{-j\frac{2\pi}{\lambda z_i}(ux+Vy)} dxdy \right\} = A\lambda z_i P\left(-\lambda z_i f_X, -\lambda z_i f_Y\right)$$

If we set $A\lambda z_i = 1$ and ignore the negative signs since for all of the interesting applications pupil functions are symmetric in x and y.

$$H\left(f_X, f_Y\right) = P\left(\lambda z_i f_X, \lambda z_i f_Y\right)$$

If the pupil function is unity within a region and zero elsewhere, then there exists a finite frequency band for which the diffraction-limitted imaging system passes all of the frequency components whithin that band without distortion. This result is for an aberration-free system.

6.2.2 Examples of amplitude transfer functions

Frequency response of diffraction-limited coherent imaging systems:

1) A square with width of $2w$

$$P(x,y) = rect\left(\frac{x}{2w}\right) rect\left(\frac{y}{2w}\right)$$

Using $H(f_X, f_Y) = P(\lambda z_i f_X, \lambda z_i f_Y)$ the amplitude transfer function is:

$$H(f_X, f_Y) = rect\left(\frac{\lambda z_i f_X}{2w}\right) rect\left(\frac{\lambda z_i f_Y}{2w}\right) = rect\left(\frac{f_X}{2f_0}\right) rect\left(\frac{f_Y}{2f_0}\right)$$

The cutoff frequency is $f_0 = \dfrac{w}{\lambda z_i}$

2) A circular aperture of radius $w = \sqrt{x^2 + y^2}$

$$P(x,y) = circ\left(\frac{\sqrt{x^2+y^2}}{w}\right)$$

$$H(f_X, f_Y) = circ\left(\frac{\sqrt{(\lambda z_i f_X)^2 + (\lambda z_i f_Y)^2}}{w}\right) = circ\left(\frac{\sqrt{f_X^2 + f_Y^2}}{w/\lambda z_i}\right) = circ\left(\frac{\sqrt{f_X^2 + f_Y^2}}{f_0}\right)$$

The cutoff frequency is $f_0 = \dfrac{w}{\lambda z_i}$

6.3 Frequency response for diffraction-limited incoherent imaging

The relationship between the pupil and amplitude transfer function in coherent case: $H\left(f_X, f_Y\right) = P\left(\lambda z_i f_X, \lambda z_i f_Y\right)$

Goal: to find the releationship between the amplitude transfer function and the system pupils for incoherent illumination. This may not be as direct a relationship as that of the coherent case.

Systems: only diffraction-limitted.

6.3.1 The optical transfer function

Incoherent (illumination) imaging systems are linear in intensity and obey the:

$$I_i(u,V) = \overbrace{\underbrace{\kappa}_{\text{Constant}} \iint_{-\infty}^{\infty} \underbrace{\left|h\left(u-\tilde{\xi}, V-\tilde{\eta}\right)\right|^2}_{\text{Intensity impulse response}} \underbrace{I_g\left(\tilde{\xi},\tilde{\eta}\right)}_{\text{Ideal image intensity}} d\tilde{\xi}d\tilde{\eta}}^{\text{Intensity convolution integral}}$$

We define the normalized frequency spectra of I_g and I_i as:

$$\mathcal{G}_g(f_X, f_Y) = \frac{\iint_{-\infty}^{\infty} I_g(u,V) e^{-j2\pi(f_X u + f_Y V)} du dV}{\underbrace{\iint_{-\infty}^{\infty} I_g(u,V) du dV}_{\text{Zero frequency value of the spectra } I_g}}$$

$$\mathcal{G}_i(f_X, f_Y) = \frac{\iint_{-\infty}^{\infty} I_i(u,V) e^{-j2\pi(f_X u + f_Y V)} du dV}{\underbrace{\iint_{-\infty}^{\infty} I_i(u,V) du dV}_{\text{Zero frequency value of the spectra } I_i}}$$

Maximum valule of the Furier transform of any real and nonnegative function $\left(\text{Such as } I_g, I_i\right)$ occures at the oringin. That maximum value is chosen as the normalization constant. This constant is related to the background light and contrast of the image.

6.3.1 The optical transfer function

We define the normalized transfer function of the system

$$\mathcal{H}(f_X, f_Y) = \frac{\iint_{-\infty}^{\infty} \left|h(u,V)\right|^2 e^{-j2\pi(f_X u + f_Y V)} du dV}{\underbrace{\iint_{-\infty}^{\infty} \left|h(u,V)\right|^2 du dV}_{\text{Zero frequency value of the spectra } h}}$$

Apply the convolution theorem to

$$I_i(u,V) = \kappa \iint_{-\infty}^{\infty} \left|h\left(u-\tilde{\xi}, V-\tilde{\eta}\right)\right|^2 I_g\left(\tilde{\xi},\tilde{\eta}\right) d\tilde{\xi} d\tilde{\eta}$$

We get: $\mathcal{G}_i(f_X, f_Y) = \mathcal{H}(f_X, f_Y)\mathcal{G}_g(f_X, f_Y)$

Based on an international agreement:

$\mathcal{H}(f_X, f_Y)$ is known as Optical Transfer Function (OTF) of the system.

$\left|\mathcal{H}(f_X, f_Y)\right|$ is known as Modulation Transfer Function (MTF) of the system that specifies the complex weighting factor applied by the system to the frequency component at (f_X, f_Y), relative to the weighting factor applied to the zero-frequency component.

6.3.1 The optical transfer function

The relationship between the OTF and MTF through h

$$\mathcal{H}(f_X, f_Y) = \mathcal{F}\{h\} \xrightarrow{\text{Normalized}} \underbrace{\mathcal{H}(f_X, f_Y)}_{\text{Incoherent}} = \frac{\mathcal{F}\left\{|h|^2\right\}}{\iint_{-\infty}^{\infty} \underbrace{|h(u,V)|}_{\text{Coherent impulse response}}{}^2 \, dudV}$$

Using the Rayleigh's (Parseval's) theorem and autocorrelation theorem

$$\iint_{-\infty}^{\infty} |h(x,y)|^2 \, dxdy = \iint_{-\infty}^{\infty} |H(f_X, f_Y)|^2 \, df_X df_Y$$

$$\mathcal{F}\left\{|h(x,y)|^2\right\} = \iint_{-\infty}^{\infty} H(\xi,\eta) H^*(\xi - f_X, \eta - f_Y) \, d\xi d\eta$$

$$\mathcal{H}(f_X, f_Y) = \frac{\iint_{-\infty}^{\infty} H(p',q') H^*(p' - f_x, q' - f_Y) \, dp'dq'}{\iint_{-\infty}^{\infty} |H(p',q')|^2 \, dp'dq'}$$

With a change of variables: $p = p' - f_X / 2, q = q' - f_Y / 2$ we get a symmetrical expression of:

$$\underbrace{\mathcal{H}(f_X, f_Y)}_{\text{OTF Incoherent system}} = \frac{\overbrace{\iint_{-\infty}^{\infty} H\left(p + \frac{f_X}{2}, q + \frac{f_Y}{2}\right) H^*\left(p - \frac{f_X}{2}, q - \frac{f_Y}{2}\right) dpdq}^{\text{Autocorrelation function of the amplitude transfer function of the coherent system}}}{\underbrace{\iint_{-\infty}^{\infty} |H(p,q)|^2 \, dpdq}_{\text{Normalization}}}$$

This equation is the <u>primary link between the coherent and incoherent systems</u> and it is valied for <u>systems with and without aberrations</u>.

6.3.2 General properties of the OTF I

Some simple and elegant properties of the OTF:

OTF is a normalized autocorrelation function.

$$\mathcal{H}\left(f_X, f_Y\right) = \frac{\int\int_{-\infty}^{\infty} H\left(p+\frac{f_X}{2}, q+\frac{f_Y}{2}\right) H^*\left(p-\frac{f_X}{2}, q-\frac{f_Y}{2}\right) dpdq}{\int\int_{-\infty}^{\infty} \left|H\left(p,q\right)\right|^2 dpdq}$$

1) $\mathcal{H}(0,0)=1$ (For proof substitute $f_X = 0,\ f_Y = 0$)

2) $\mathcal{H}(-f_X, -f_Y) = \mathcal{H}^*(f_X, f_Y)$ (For proof use "Fourier transform of a real function has Hermitian symmetry")

3) $\left|\mathcal{H}(f_X, f_Y)\right| \leq \mathcal{H}(0,0)$ (For proof use the Schwarz's inequality)

If $X(p,q)$ and $Y(p,q)$ are any two complex valued functions of (p,q), then

$$\left|\int\int XYdpdq\right|^2 \leq \int\int |X|^2 dpdq \int\int |Y|^2 dpdq$$

The equality holds ony if $Y = KX^*$ where K is a complex constant.

6.3.2 General properties of the OTF II

Now let $X\left(p,q\right)=H\left(p+\frac{f_X}{2},q+\frac{f_Y}{2}\right)$ and $Y\left(p,q\right)=H^*\left(p-\frac{f_X}{2},q-\frac{f_Y}{2}\right)$

Then we find

$$\left|\iint_{-\infty}^{\infty} H\left(p+\frac{f_X}{2},q+\frac{f_Y}{2}\right)H^*\left(p-\frac{f_X}{2},q-\frac{f_Y}{2}\right)dpdq\right|^2$$

$$\leq \iint_{-\infty}^{\infty}\left|H\left(p+\frac{f_X}{2},q+\frac{f_Y}{2}\right)\right|^2 dpdq \iint_{-\infty}^{\infty}\left|H^*\left(p-\frac{f_X}{2},q-\frac{f_Y}{2}\right)\right|^2 dpdq$$

$$=\left[\iint_{-\infty}^{\infty}\left|H\left(p,q\right)\right|^2 dpdq\right]^2$$

$$\frac{\left|\iint_{-\infty}^{\infty} H\left(p+\frac{f_X}{2},q+\frac{f_Y}{2}\right)H^*\left(p-\frac{f_X}{2},q-\frac{f_Y}{2}\right)dpdq\right|}{\left|\iint_{-\infty}^{\infty}\left|H\left(p,q\right)\right|^2 dpdq\right|}\leq 1 \text{ therefore}$$

$\left|\mathcal{H}\left(f_X,f_Y\right)\right|\leq 1$ using the property (1) $\mathcal{H}\left(0,0\right)=1$ we get

$\left|\mathcal{H}\left(f_X,f_Y\right)\right|\leq\left|\mathcal{H}\left(0,0\right)\right|$

Absolute intensity of the image background is never the sam as the absolute intensity of the object background. The normalization has removed the information about the absolute intensity levels.

6.3.3 The OTF of an aberration-free system I

Up to this point we have not made any assumptions about the aberration of the system.

Now we consider the <u>diffraction-llimitted incoherent system</u>:

We have $H\left(f_X, f_Y\right) = P\left(\lambda z_i f_X, \lambda z_i f_Y\right)$ for coherent systems

For an incoherent system, it follows from the equation for OTF with a change of variables as: $f_X = \lambda z_i f_X,\ f_Y = \lambda z_i f_Y$

$$\mathcal{H}\left(f_X, f_Y\right) = \frac{\iint_{-\infty}^{\infty} P\left(x + \frac{\lambda z_i f_X}{2}, y + \frac{\lambda z_i f_Y}{2}\right) P\left(x - \frac{\lambda z_i f_X}{2}, y - \frac{\lambda z_i f_Y}{2}\right) dxdy}{\iint_{-\infty}^{\infty} P\left(x, y\right) dxdy}$$

We used $\left|P\left(x, y\right)\right|^2 = P\left(x, y\right)$ since $P\left(x, y\right)$ is equal to zero or unity.

6.3.3 The OTF of an aberration-free system II

The numinator of the OTF of a diffraction-limitted incoherent system represents the area of overlap of two displaced pupil functions:

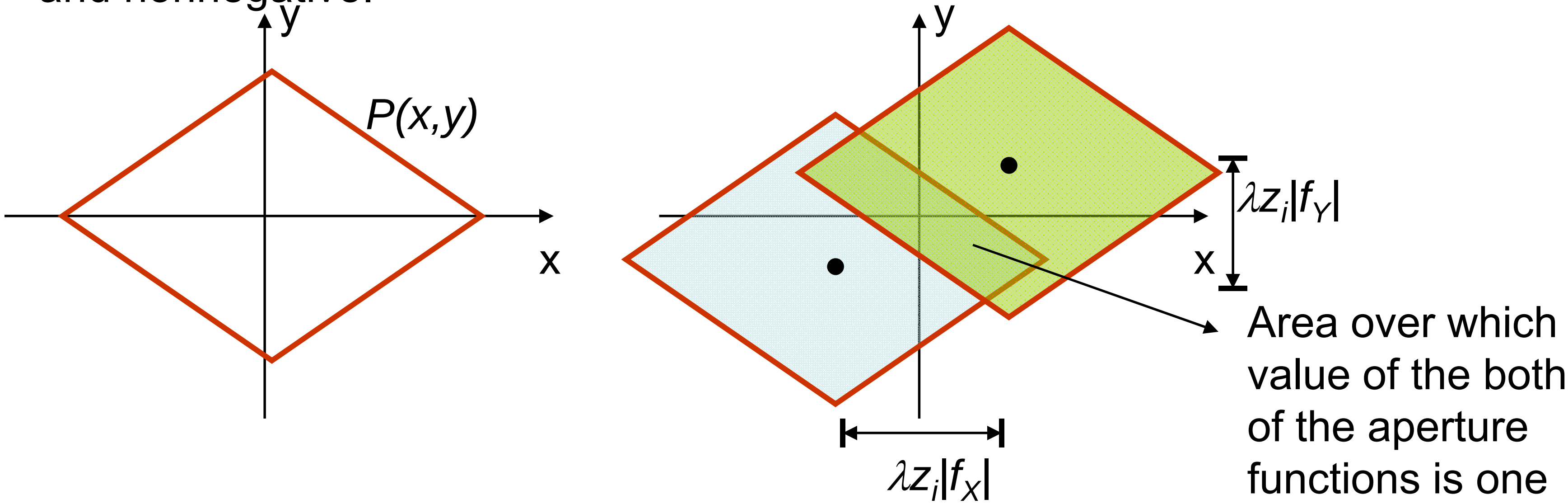
1) One centered at $\left(\lambda z_i f_X / 2,\ \lambda z_i f_Y / 2\right)$

2) Second centered at dimetrically opposite point $\left(-\lambda z_i f_X / 2, -\lambda z_i f_Y / 2\right)$

The denuminator normalizes the area of overlap by the total area of the pupil.

Thus $\mathcal{H}(f_X, f_Y) = \dfrac{\text{area of overlap}}{\text{total area}}$

Figure shows a geometrical optics interpretation of the OTF of a diffraction-limitted incoherent system. This interpretation implies that OTF is always real and nonnegative.



6.3.3 The OTF of an aberration-free system III

$$\mathcal{H}(f_X, f_Y) = \frac{\iint_{-\infty}^{\infty} P\left(x + \frac{\lambda z_i f_X}{2}, y + \frac{\lambda z_i f_Y}{2}\right) P\left(x - \frac{\lambda z_i f_X}{2}, y - \frac{\lambda z_i f_Y}{2}\right) dxdy}{\iint_{-\infty}^{\infty} P(x, y)\, dxdy}$$

Computational appraoch to calculation of the OTF of a complicated diffraction-limitted system:

1) Inverse Fourier transform the reflected pupil function $P(-x,-y)$ or Fourier transform the pupil function $P(x,y)$, to find the amplitude point-spread function.

2) Take the squared magnitude of the amplitude point-spread function to find the Intensity point-spread function.

3) Take the Fourier transform of the Intensity point-spread function to find the unnormalized OTF.

4) Normalize the OTF to unity at the origin.

6.3.3 The OTF of an aberration-free system IV

$$\mathcal{H}(f_X, f_Y) = \frac{\iint_{-\infty}^{\infty} P\left(x+\frac{\lambda z_i f_X}{2}, y+\frac{\lambda z_i f_Y}{2}\right) P\left(x-\frac{\lambda z_i f_X}{2}, y-\frac{\lambda z_i f_Y}{2}\right) dxdy}{\iint_{-\infty}^{\infty} P(x,y)\,dxdy}$$

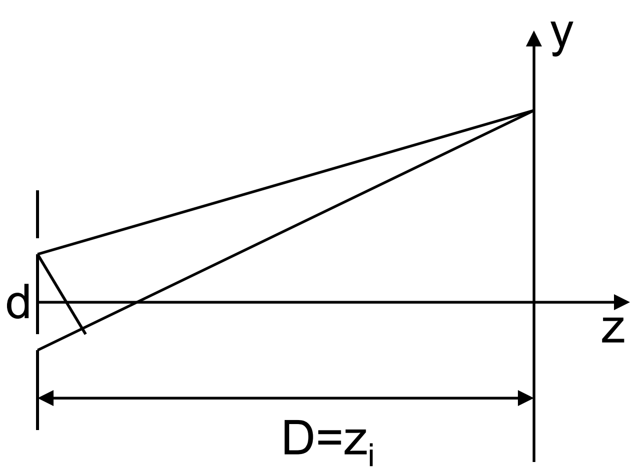
Question: how a sinusoidal componet at a particular frequency pair (f_X, f_Y) is generated?

Answere: one way is by interference of light in the image plane from two separate patches on the exit pupil of the system with separation $(\lambda z_i \mid f_X \mid, \lambda z_i \mid f_Y \mid)$.

There is more than one pair of patches that are separated by $(\lambda z_i \mid f_X \mid, \lambda z_i \mid f_Y \mid)$.

Weight of each frequency component is determined by the number of ways the corresponding separation can be fit into the pupil.

The number of ways a particular separation can be fit into the exit pupil is proportional to the area of overlap of pupils separated by this spacing.



$$d \sin\theta = m\lambda$$

$$\tan\theta = |y| / z_i$$

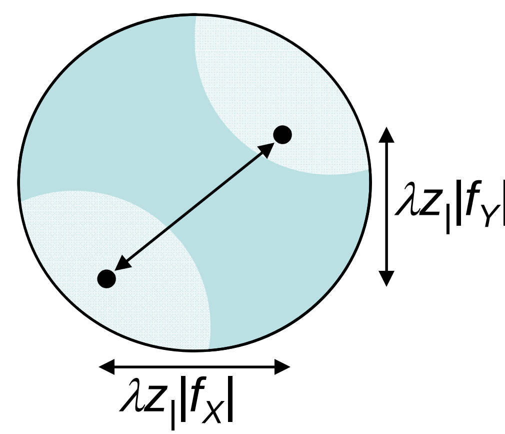
$$|y| \approx \lambda z_i m / d$$

$$|y_{m+1} - y_m| = |\lambda_X| = \lambda z_i / d$$

$$|f_X| = |1/\lambda_X| = d / \lambda z_i$$

$$d = |f_X| \lambda z_i$$

Separation of the patches that light is coming from to produce fringes of freq. $|f_X|$



6.3.4 Examples of diffraction-limited OTFs

1) OTF of a system with square pupil of width $2w$. We need to calculate the area of overlap between two pupils separated by $(\lambda z_i |f_X|, \lambda z_i |f_Y|)$ centered at $(\lambda z_i f_X/2, \lambda z_i f_y/2)$ and $(-\lambda z_i f_X/2, -\lambda z_i f_y/2)$.

$$A(f_X, f_Y) = \begin{cases} (2w - \lambda z_i |f_X|)(2w - \lambda z_i |f_Y|) & |f_X| \le \frac{2w}{\lambda z_i}, |f_Y| \le \frac{2w}{\lambda z_i} \\ 0 & \text{otherwise} \end{cases}$$

Normalizing this area with the total area of $4w^2$ we get:

$$\mathcal{H}(f_X, f_Y) = \begin{cases} \frac{(2w - \lambda z_i |f_X|)}{2w} \frac{(2w - \lambda z_i |f_Y|)}{2w} & |f_X| \le \frac{2w}{\lambda z_i}, |f_Y| \le \frac{2w}{\lambda z_i} \\ 0 & \text{otherwise} \end{cases}$$

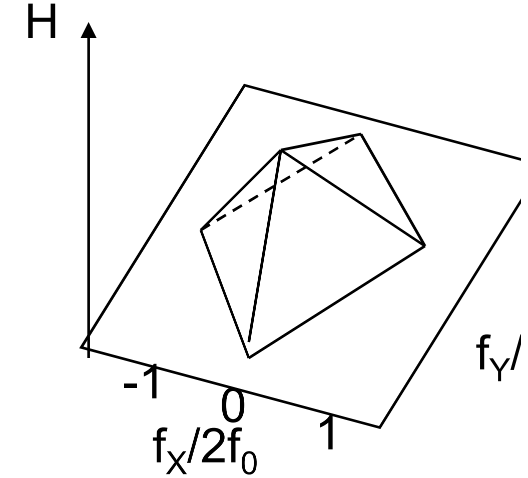
$$\mathcal{H}(f_X, f_Y) = \begin{cases} \left(1 - \frac{f_X}{2f_0}\right)\left(1 - \frac{f_Y}{2f_0}\right) & |f_X| \le \frac{2w}{\lambda z_i}, |f_Y| \le \frac{2w}{\lambda z_i} \\ 0 & \text{otherwise} \end{cases}$$

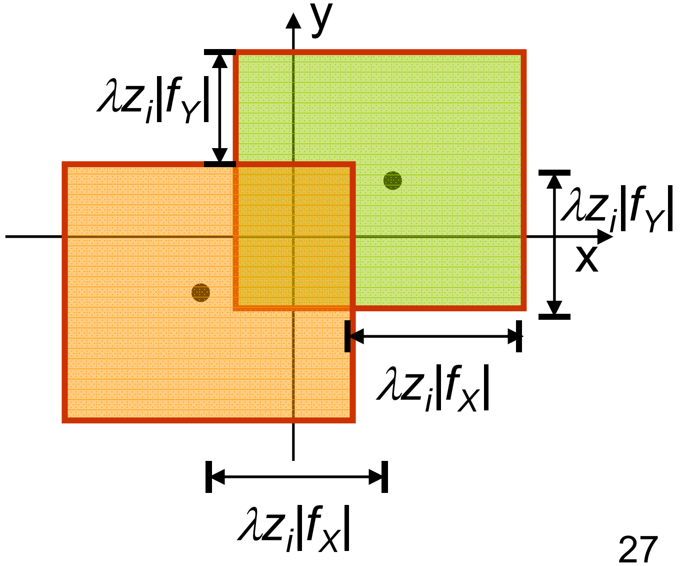
Where $f_0 = \frac{w}{\lambda z_i}$

$$\mathcal{H}(f_X, f_Y) = \Lambda\left(\frac{f_X}{2f_0}\right)\Lambda\left(\frac{f_Y}{2f_0}\right)$$

Cutoff frequency: $2f_0 = \frac{2w}{\lambda z_i}$

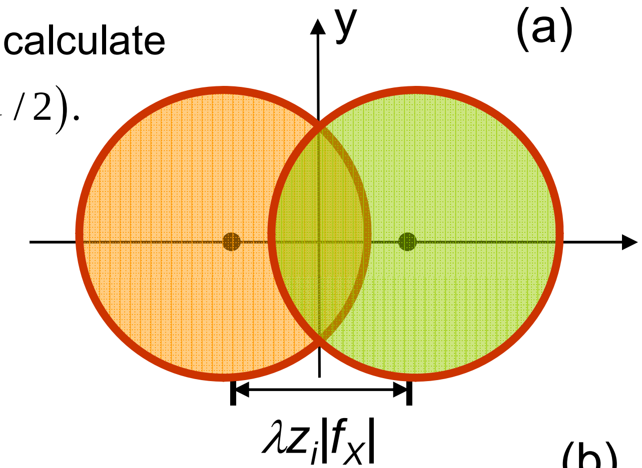
In this case OTF extends twice the cutoff frequency of the coherent system. Compare Fig. 6.7 and 6.3





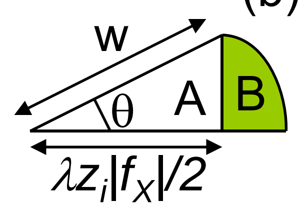
6.3.4 Examples of diffraction-limited OTFs

2) OTF of a system with circular pupil of diameter $2w$. We need to calculate the area of overlap between two pupils centered at $\left(\lambda z_i f_X/2, \lambda z_i f_Y/2\right)$.



It is not as simple as the square case. We know the OTF will be circularly symmetric. So we calculate $\mathcal{H}$ along the x axis and then rotate that around the center of it.

The shaded area in (a) is four times the B area in (b). First get B



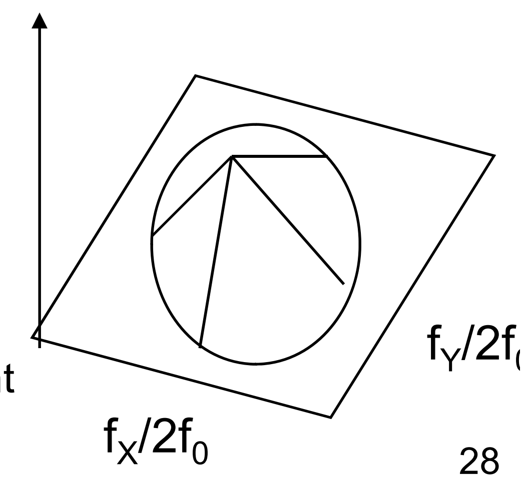
$$Area(A+B)=\left[\frac{\theta}{2\pi}\right]\left(\pi w^2\right)=\left[\frac{\cos^{-1}\left(\lambda z_i f_X/2w\right)}{2\pi}\right]\left(\pi w^2\right)$$

$$Area(A)=\frac{1}{2}\left(\frac{\lambda z_i f_X}{2}\right)\sqrt{w^2-\left(\frac{\lambda z_i f_X}{2}\right)^2}$$

$$\mathcal{H}(f_X,0)=\frac{4\left[area(A+B)-area(A)\right]}{\pi w^2}=\frac{\left[\frac{\cos^{-1}\left(\lambda z_i f_X/2w\right)}{2\pi}\right]\left(\pi w^2\right)-\frac{1}{2}\left(\frac{\lambda z_i f_X}{2}\right)\sqrt{w^2-\left(\frac{\lambda z_i f_X}{2}\right)^2}}{\pi w^2}$$

For general radial distance ρ in the frequency plane

$$\mathcal{H}(\rho)=\begin{cases}\frac{2}{\pi}\left[\cos^{-1}\left(\frac{\rho}{2\rho_0}\right)-\frac{\rho}{2\rho_0}\sqrt{1-\left(\frac{\rho}{2\rho_0}\right)^2}\right] & \rho\le 2\rho_0 \text{ ,where } \rho_0=\frac{w}{\lambda z_i}\\ 0 & \text{otherwise}\end{cases}$$



In this case again OTF extends twice the cutoff frequency of the coherent system. Compare Fig. 6.9 and 6.3

6.4 Aberrations and their effects on frequency response

- For diffraction-limited systems we assumed that a point source object yields at the exit pupil a perfect spherical wavefront converging towards the ideal geometrical image point.
- For systems with aberrations the exit pupil wavefront departs from a perfect sphere.
- We do not treat the aberration subject completely.
- We will concentrate on general effects of aberration on the frequency response of a system.
- We will show these effects with a simple example.

6.4.1 The generalized pupil function I

For a diffraction limited system, amplitude point-spread function is the Fraunhofer diffraction pattern of the exit pupil centered on the ideal image point.

To include effects of the aberrations in this picture we assume:

1) the pupil is still illuminated by a perfect spherical wave

2) there is a phase mask in the aperture deforming the wavefront that leaves the pupil.

3) $kW(x,y)$ represents effect of the phase error at point (x,y) where $k = 2\pi/\lambda$ and $W(x,y)$ is an effective path length error or the aberration function .

4) The complex amplitude transmittanc of the imaginary phase shift plane is: $\mathcal{P}(x,y) = P(x,y)e^{jkW(x,y)}$

the complex function $\mathcal{P}(x,y)$ is the generalized pupil function.

6.4.1 The generalized pupil function II

The generalized pupil function defined as: $\mathcal{P}(x, y) = P(x, y)e^{jkW(x,y)}$

Now with the above assumptions

a) The amplitude transmittance function of an aberrated coherent system is the Fraunhofer diffraction pattern of the generalized pupil function $\mathcal{P}(x, y)$.

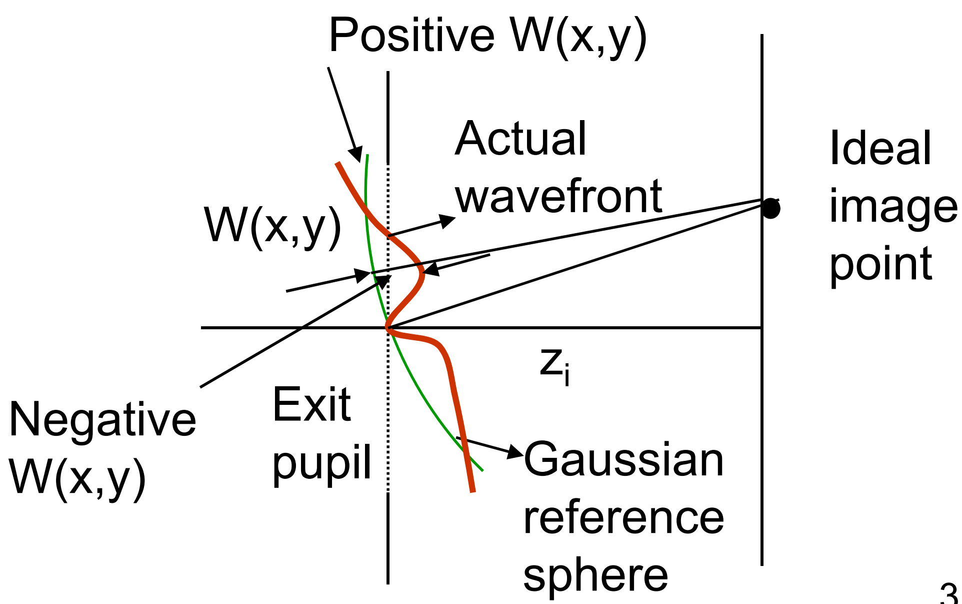
b) The intensity impulse response of an aberrated incoherent system is the squared magnitude of the amplitude impulse response.

c) Aberration function is defined with respect to a Gaussian reference sphere.

d) Gaussian reference sphere is the ideal spherical surface centered at the ideal image point and passing through the point where the optical axis pierces the exit pupil.

e) The actual wavefront is also defined to intercept the optical axis in the exit pupil.

f) The error caused by aberration function can be negative or positive.



6.4.2 Effects of aberrations on the amplitude transfer function

For a diffraction-limited coherent system:

1) impulse response is Fourier transform of the pupil function.

$$h(u,V)=\frac{A}{\lambda z_i}\iint_{-\infty}^{\infty}P(x,y)e^{-j\frac{2\pi}{\lambda z_i}(ux+Vy)}dxdy$$

2) The amplitude transfer function is the Fourier transform of the amplitude impulse response $H(f_X,f_Y)=\iint_{-\infty}^{\infty}h(u,V)e^{-j2\pi(f_Xu+f_YV)}dudV$

3) Therefore the amplitude transfer function was found to be proportional to the scaled pupil function $H(f_X,f_Y)=P(\lambda z_i f_X,\lambda z_i f_Y)$

For diffraction limited coherent systems with aberration:

a) The generalized pupil function is: $\mathcal{P}(x,y)=P(x,y)e^{jkW(x,y)}$

b) The amplitude transfer function is written as:

$$H(f_X,f_Y)=\mathcal{P}(\lambda z_i f_X,\lambda z_i f_Y)=P(x,y)e^{jkW(\lambda z_i f_X,\lambda z_i f_Y)}$$

c) The band-limitation of the $H(f_X,f_Y)$ that was imposed by the finite exit pupil has not changed in presence of the aberrations.

d) Aberrations only introduce phase distortions within the passband.

6.4.3 Effects of aberrations on the OTF (incoherent systems) I

OTF of an aberration-free incoherent diffraction-limited system:

$$\mathcal{H}(f_X, f_Y) = \frac{\iint_{-\infty}^{\infty} P\left(x+\frac{\lambda z_i f_X}{2}, y+\frac{\lambda z_i f_Y}{2}\right) P\left(x-\frac{\lambda z_i f_X}{2}, y-\frac{\lambda z_i f_Y}{2}\right) dxdy}{\iint_{-\infty}^{\infty} P(x,y)\, dxdy}$$

We define the function $\mathcal{A}(f_X, f_Y)$ as the area of overlap of

$P\left(x+\frac{\lambda z_i f_X}{2}, y+\frac{\lambda z_i f_Y}{2}\right)$ and $P\left(x-\frac{\lambda z_i f_X}{2}, y-\frac{\lambda z_i f_Y}{2}\right)$

then $\mathcal{H}(f_X, f_Y) = \dfrac{\iint_{\mathcal{A}(f_X, f_Y)} dxdy}{\iint_{\mathcal{A}(0,0)} dxdy}$

In presence of aberrations: $\overbrace{\mathcal{H}(f_X, f_Y)}^{\text{OTF}} = \dfrac{\iint_{\mathcal{A}(f_X, f_Y)} e^{\overbrace{jk\left[W\left(x+\frac{\lambda z_i f_X}{2}, y+\frac{\lambda z_i f_Y}{2}\right) - W\left(x-\frac{\lambda z_i f_X}{2}, y-\frac{\lambda z_i f_Y}{2}\right)\right]}^{\text{Wavefront errors caused by aberrations}}} dxdy}{\iint_{\mathcal{A}(0,0)} dxdy}$

Aberrations will never increase the $MTF = \left|\mathcal{H}(f_X, f_Y)\right|$ or the modulus of the OTF.

6.4.3 Effects of aberrations on the OTF (incoherent systems) II

Important conclusions in presence of aberrations:

1) $\left|\mathcal{H}\left(f_X , f_Y \right)\right|^2_{\text{With aberrations}} \leq \left|\mathcal{H}\left(f_X , f_Y \right)\right|^2_{\text{Without aberrations}}$

implies that aberrations <u>can never increase contrast</u> of any spatial frequency component of the image.

2) The absolute cutoff frequency of the system remains thae same however, severe aberrations can reduce high frequency portions of OTF to an extent that the <u>effective cutoff is much lower</u> than the diffraction-limited cutoff.

3) For certian frequency bands OTF may have negative or even complex value. <u>when OTF is negative</u>, the image components at that frequency can undergo a <u>contrast reversal</u>; i.e. <u>intensity peaks become intentity nulls</u> and vice versa.

6.4.4 Examples of a simple aberration: a focusing error

Most aberrations are mathematically very challenging subjecst.

The simplest example is a focusing error for an square aperture system:

The center of curvature of a spherical wavefront converging towards image of a point-source object is either to the left or right of the image plane.

For simplicity we assume this point is still on the optical axis.

Ideal phase distribution across the exit pupil: $\phi_i(x,y) = -\frac{\pi}{\lambda z_i}(x^2+y^2)$

Actual phase distribution across the exit pupil: $\phi_a(x,y) = -\frac{\pi}{\lambda z_a}(x^2+y^2)$

where $z_a \neq z_i$

The path length error or aberration function: $kW(x,y) = \phi_a(x,y) - \phi_i(x,y)$

$$kW(x,y) = -\frac{\pi}{\lambda z_a}(x^2+y^2) + \frac{\pi}{\lambda z_i}(x^2+y^2) \rightarrow W(x,y) = -\frac{1}{2}\left(\frac{1}{z_a} - \frac{1}{z_i}\right)\underbrace{(x^2+y^2)}_{\text{Quadratic dependence on the space variables on the exit pupil}}$$

For a square aperture of width $2w$ the maximum path-length difference at the edge of the aperture along the x or y axis is: $W_m = -\frac{1}{2}\left(\frac{1}{z_a} - \frac{1}{z_i}\right)w^2$

$$W(x,y) = W_m \frac{x^2+y^2}{w^2}$$

6.4.4 Examples of a simple aberration: a focusing error

Now substitute the path-length difference $W(x,y) = W_m \frac{x^2+y^2}{w^2}$ in the OTF:

$$\mathcal{H}(f_X, f_Y) = \frac{\iint_{\mathcal{A}(f_X, f_Y)} e^{jk\left[W\left(x+\frac{\lambda z_i f_X}{2}, y+\frac{\lambda z_i f_Y}{2}\right) - W\left(x-\frac{\lambda z_i f_X}{2}, y-\frac{\lambda z_i f_Y}{2}\right)\right]} dxdy}{\iint_{\mathcal{A}(0,0)} dxdy}$$

$$\mathcal{H}(f_X, f_Y) = \frac{\iint_{\mathcal{A}(f_X, f_Y)} e^{jk\frac{W_m}{w^2}\left[\left(x+\frac{\lambda z_i f_X}{2}\right)^2 + \left(y+\frac{\lambda z_i f_Y}{2}\right)^2 - \left(x-\frac{\lambda z_i f_X}{2}\right)^2 + \left(y-\frac{\lambda z_i f_Y}{2}\right)^2\right]} dxdy}{\iint_{\mathcal{A}(0,0)} dxdy}$$ see page 144 eq 6.31

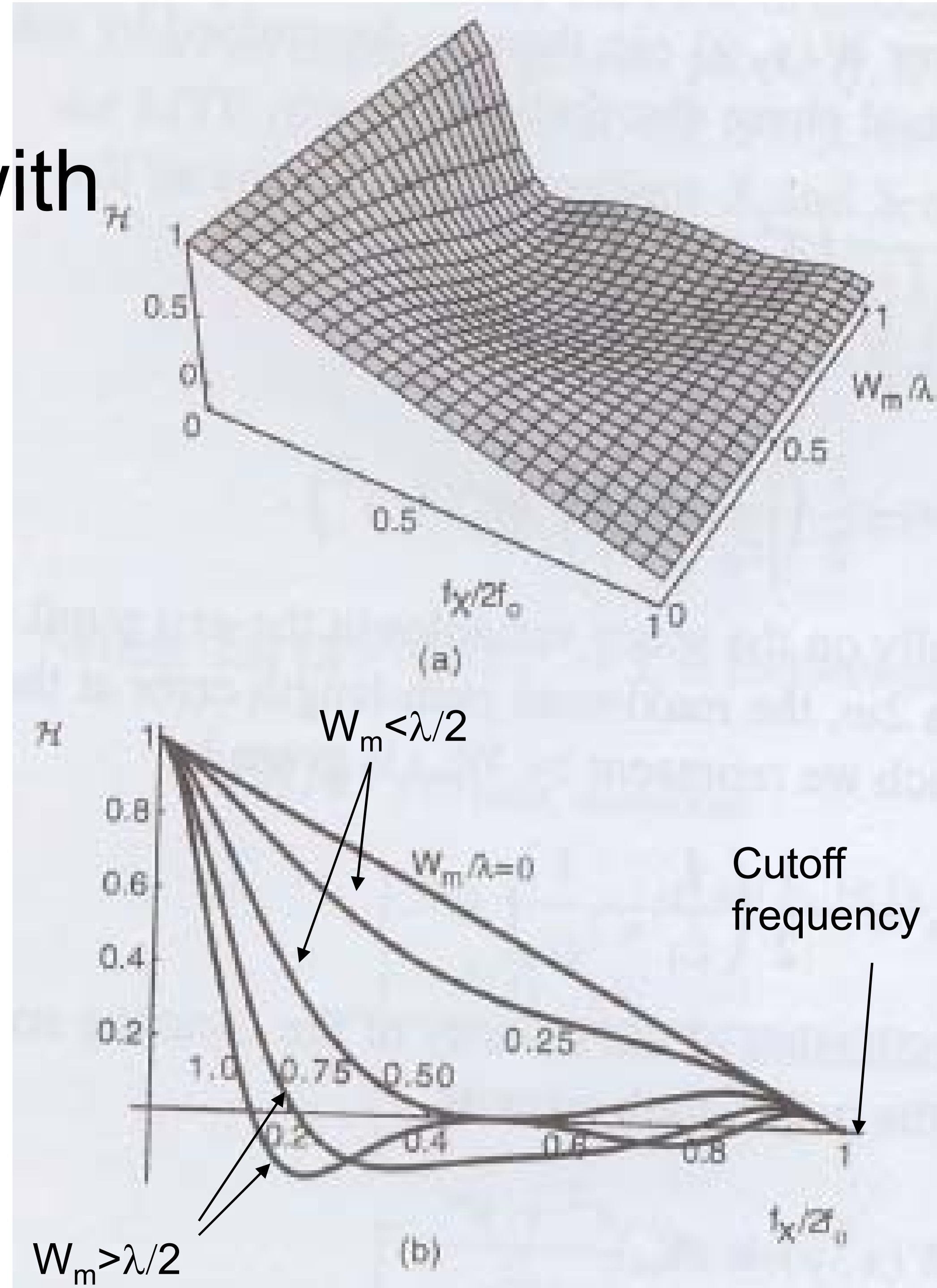
$$\mathcal{H}(f_X, f_Y) = \Lambda\left(\frac{f_X}{2f_0}\right)\Lambda\left(\frac{f_Y}{2f_0}\right)\sin c\left[\frac{8W_m}{\lambda}\left(\frac{f_X}{2f_0}\right)\left(1-\frac{|f_X|}{2f_0}\right)\right]\sin c\left[\frac{8W_m}{\lambda}\left(\frac{f_Y}{2f_0}\right)\left(1-\frac{|f_Y|}{2f_0}\right)\right]$$

We plot this OTF for various values of W_m/λ

$W_m = 0$ gives the diffration-limited $($no aberration$)$ OTF.

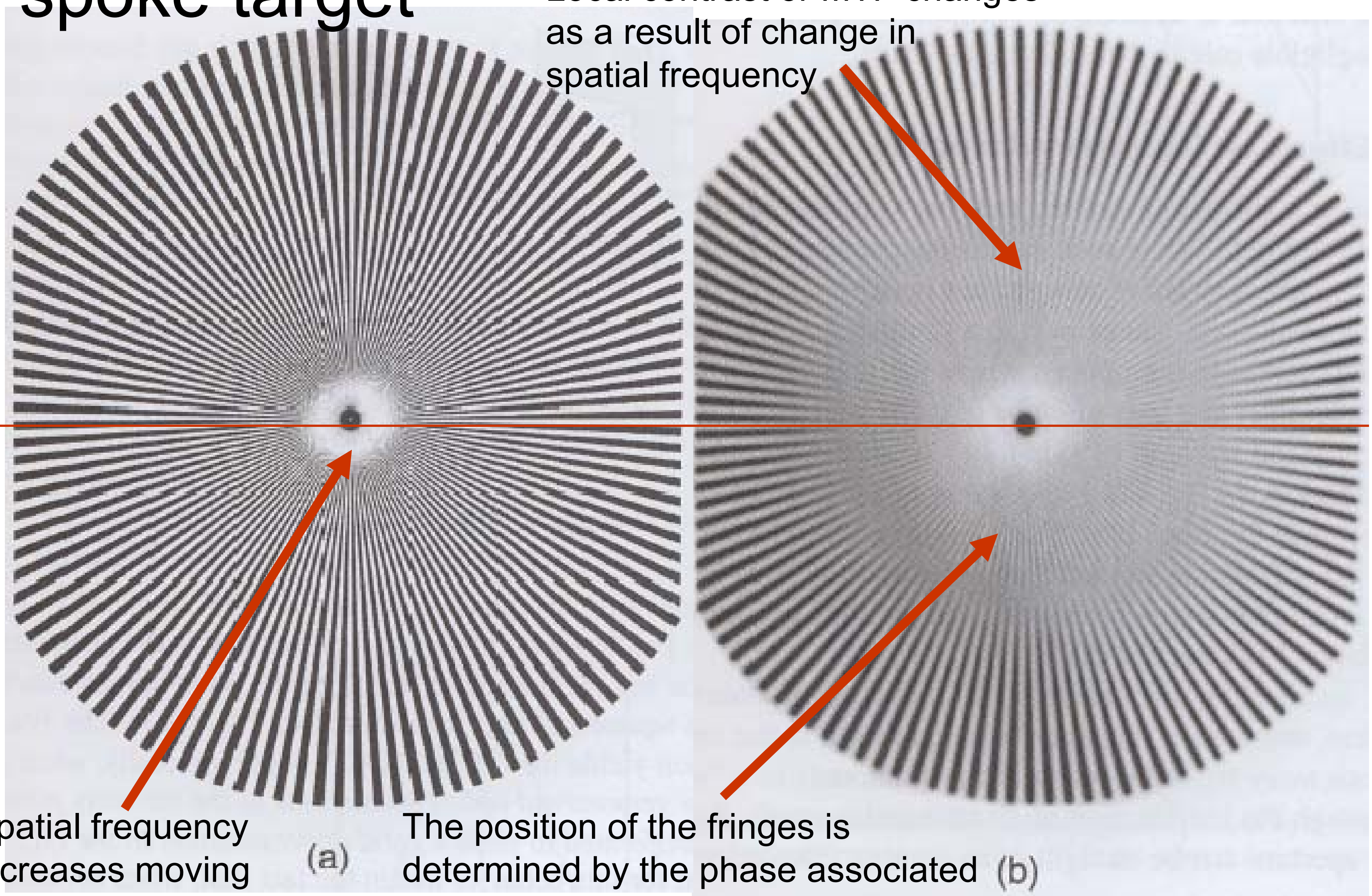
For $W_m > \lambda/2$ sign of the OTF is reversed meaning a contrast reversal.

OTF with focusing error in a system with square aperture



Focused and misfocused images of a spoke target

Local contrast or MTF changes as a result of change in spatial frequency



Spatial frequency increases moving to center

The position of the fringes is determined by the phase associated with OTF at each frequency

6.4.4 Examples of a simple aberration: a focusing error

Consider the form of OTFwhen the focusing error is very sever $W_m >> \lambda/2$

$$\mathcal{H}(f_X, f_Y) = \Lambda\left(\frac{f_X}{2f_0}\right)\Lambda\left(\frac{f_Y}{2f_0}\right)\sin c\left[\frac{8W_m}{\lambda}\left(\frac{f_X}{2f_0}\right)\left(1-\frac{|f_X|}{2f_0}\right)\right]\sin c\left[\frac{8W_m}{\lambda}\left(\frac{f_Y}{2f_0}\right)\left(1-\frac{|f_Y|}{2f_0}\right)\right]$$

In this case value of $\mathcal{H}$ is negligable over large frequencies and only it is nonzero for

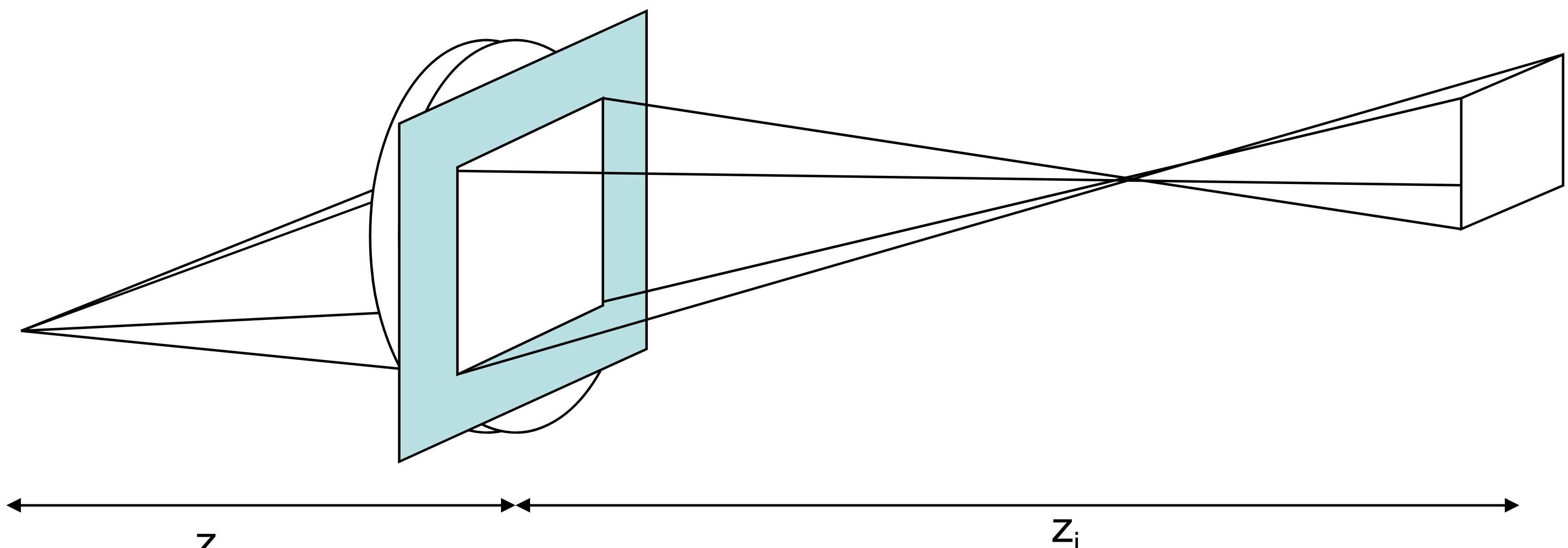
small $\frac{|f_X|}{2f_0}$ so $\frac{|f_X|}{2f_0}$<<1 and $1-\frac{|f_X|}{2f_0}\approx 1$ and $1-\frac{|f_Y|}{2f_0}\approx 1$

$\Lambda\left(\frac{f_X}{2f_0}\right)\approx\Lambda\left(\frac{f_Y}{2f_0}\right)\approx 1$ for $\frac{|f_X|}{2f_0}$<<1

$$\mathcal{H}(f_X, f_Y) = \sin c\left[\frac{8W_m}{\lambda}\left(\frac{f_X}{2f_0}\right)\right]\sin c\left[\frac{8W_m}{\lambda}\left(\frac{f_Y}{2f_0}\right)\right]$$

This is precisely the OTF predicted by the geometrical optics saying that the point-spread function of the system is going to be the geometrical projection of the exit pupil into the image plane.

Geometrical optics prediction of the point-spread function of a system with square aperture and sever focusing error



Fourier transform of such a PSF function gives the OTF of the system in presence of severe aberrations, Fourier transform of the geometrical PSF is a good approximation for the OTF of the system and diffraction plays a negligible role is shape of the image

6.4.5 Apodization and its effects on Frequency response