Light Review

Wave motion

Maxwell equations

Light as an electromagnetic wave

Energy and momentum

Eradat Spring 2010 SJSU

References: R1 & R2 by Hecht

Optics

- Study of light
  - In wave optics or physical optics light is an electromagnetic wave. Example phenomena: diffraction, interference,…
  - In geometrical optics light is a ray. Example: refraction, reflection, …
  - In quantum optics light is a particle. Example: absorption, emission, laser action, …

Wave

- A self sustaining energy-carrying <u>disturbance</u> of a medium through which it propagates.
  - **Longitudinal wave**: the medium is displaced in the direction of motion of the wave.
  - **Transverse wave**: the medium is displaced in a direction perpendicular to that of the motion of the wave.
- When a wave propagates, the disturbance advances, <u>not the medium</u>. That is why waves can propagate faster than the medium carrying them (Leonardo da Vinci).

Mathematical expression of a wave

The most general form of a one-dimentional wavefunction is $\psi(x,t) = f(x,t)$ where ψ is a disturbance and f is the shape of it.

If we choose a function of x (a shape) and let $x \rightarrow x\text{-}Vt$ then we have a wavefunction moving in $+x$ direction with speed of V.

It is also common to express a wave as a function of $(t - \dfrac{x}{V})$:

$$f(x - Vt) = F\left(\frac{x - Vt}{-V}\right) = F\left(t - \frac{x}{V}\right)$$

Exercise

1.1) Use MATLAB or equivalent software for plotting.

a) Make a bell-shaped wavefunction moving in $+x$ diraction.

b) Make a wavefunction moving in $-x$ from the following pulse $f(x) = 3/(10x^2 + 1)$.

c) Plot both wavefunctions at $t = -2,\ -1,\ 0,\ 1,\ 2,\ 3s$

Ans. a) Bell shape or the Gaussian function $f(x) = e^{-ax^2}, \psi(x,t) = e^{-a(x-Vt)^2}$

b) $\psi(x,t) = \dfrac{3}{10(x+Vt)^2 + 1}$

Note

- One-dimensional (1D) wave: direction of propagation is a function of one space variable. Example: disturbance in a rope.
- Two-dimensional (2D) wave: direction of propagation is a function of two space variables. Example: ripples in a pond (circular).

Wave equation

- Maxwell showed that light is a transverse wave with electric and magnetic fields varying in directions perpendicular to the direction of propagation. It can be expressed by a second order differential equation.
- In 1D, the wave equation is:

$$\frac{\partial^2 \psi(x,t)}{\partial x^2} = \frac{1}{V^2} \frac{\partial^2 \psi(x,t)}{\partial t^2}$$

- $\Psi(x, t)$ is the wavefunction representing the disturbance as a function of space and time. V is the speed of the disturbance.
- This is a homogeneous equation means it does not contain ψ by itself; equivalent to lack of driving source.
- Un-damped means it does not contain a first order time derivative of ψ; equivalent to lack of resistance.
- It is a linear equation since all derivatives appear in first power.

Exercise

1.2) Show that $\psi(x.t) = f(x \mp Vt)$ is a solution to the one-dimentional differential wave equation $\frac{\partial^2 \psi(x,t)}{\partial x^2} = \frac{1}{V^2}\frac{\partial^2 \psi(x,t)}{\partial t^2}$. What kind of differential equation is this? How many constants do we need to express $\psi(x,t)$ uniquely? Define the constants*.

1.3) superposition priciple for linear diff. equations: If $\psi_1(x,t)$ and $\psi_2(x,t)$ are both solutions of the differential wave equation, show that $a\psi_1(x,t) + b\psi_2(x,t)$ is also a solution where a and b are constants.

1.4) Show that the expression $\psi(x,t) = Ae^{-4(x+\frac{3}{2}t)^2}$ is a progressive wave that is solution of the wave equation. What is the velocity of this wave (magnitude and direction)?

Answers

1.2) take derivative of ψ with respect to x and t twice and insert into the differential equation. This is a linear second order differential equation. Since the equation is second order we need two constants which are amplitude and frequency.

1.3) Take derivative of the $a\psi_1 + b\psi_2$ and insert into the wave equation.

1.4) Need to show that the expression is a solution to the wave equation with $V = -3/2$ in negative x direction.

Waves in three dimension

The one-dimensional wave equation can be written in there-dimensions if:

a) all three space variables appear symmetrically

b) by interchanging the space variables the equation do not change (right or left-handedness of the system has to remain the same). Wave equation in Cartezian coordinates:

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}+\frac{\partial^2\psi}{\partial z^2}=\frac{1}{V^2}\frac{\partial^2\psi}{\partial t^2};\ \text{ with } \nabla^2\equiv\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2},$$

We have the concise form of the <u>3D wave equation</u>:

$$\boxed{\nabla^2\psi=\frac{1}{V^2}\frac{\partial^2\psi}{\partial t^2}}$$

Harmonic waves I

A wave with sinusoidal profile is said to be harmonic.

$\psi(x,0) = A\sin kx$

A progressive wave function made of such a profile is then

$\psi(x,t) = A\sin k(x \mp Vt)$

Why we need k the propagation constant? Argument of a $\sin$ function has to be unitless so the positive constant k provides that condition $[k] = 1/L$.

Two conditons for having harmonic waves:

The wavefuction has to repeat itself after a spatial period λ.

The wavefuction has to repeat itself after a temporal period τ.

Harmonic waves II

Period: the number of units of time per one wave or $\tau = \dfrac{x}{V} = \dfrac{\lambda}{V}$

Frequency: reciprocal of period or number of waves per unit time

or $\nu = f = \dfrac{V}{\lambda}$

Amplitude: the maximum value of the disturbance ψ.

Since $-1 \le \sin kx \le 1$, the amplitude is equal to A

Angular frequency: $\omega = 2\pi\nu$ ($\Delta\omega = 2\pi$ radian is equal to one cycle)

A harmonic wavefunction, $\psi(x,t) = A\sin(kx \mp \omega t)$ ranges from $-\infty$ to $+\infty$ in space and time and it is a mathematical idealization for a truly monochromatic or single frequency wave.

In reality we only have quasi-monochromatic waves.

Exercise

1.5) Show that for a harmonic wave $k = 2\pi / \lambda$

1.6) Show that for a harmonic wave $\tau = \lambda / V$

1.7) Sketch the wave $\psi(x,t) = A\sin(kx - \omega t)$ at $t = 0,\ \tau/4,$ $\tau/2,\ 3\tau/4, \tau$. Use MATLAB to plot the function at these times.

1.8) Verify that the harmonic wavefunction is a solution of the 1-dimensional differential wave equation.

1.9) Prove that the following expressions are all alternatives for the harmonic wavefunction $\psi = A\sin(kx \mp \omega t)$

$$\psi = A\sin 2\pi\left(\frac{x}{\lambda} \mp \frac{t}{\tau}\right) = A\sin 2\pi\nu\left(\frac{x}{V} \mp t\right) = A\sin 2\pi(\kappa x \mp \nu t);$$

Where V is velocity of the wave and $\kappa = 1/\lambda$ is the wavenumber.

Phase of a wave

Phase of a harmonic wave: the argument of the sin function.

Initial phase: value of the phase at $t = 0,\ x = 0.$

If $\phi = 0$ at this point then the function has zero initial phase.

For the wavefunction $\psi = A\sin(kx \mp \omega t)$ what is the phase and the initial phase?

What is the initial phase of the folowing wavefunction?

$\psi = A\sin(kx \mp \omega t + \varepsilon)$?

Phase velocity of a wave

Constant-phase point: a point on a progressive wave with a constant magnitude of the disturbance.

$$\psi(x,t) = A\sin(kx \mp \omega t + \varepsilon)$$

Phase velocity of a wave is speed of the motion of a constant-phase point on a disturbance $V_{phase} = \left.\frac{\partial x}{\partial t}\right|_{\phi}$

Phase velocity is speed of the motion of the disturbance.

Ex: calculate value of the phase velocity for the above wave.

Phase of a harmonic wave at time t: $\phi = kx \mp \omega t + \varepsilon$

For constant phase: $\frac{\partial \phi}{\partial t} = 0 \rightarrow k \left.\frac{\partial x}{\partial t}\right|_{\phi} \mp \omega = 0 \rightarrow \boxed{V_{phase} == \pm \frac{\omega}{k}}$

Exercise

1.10) For the wavefunction $\psi(x,t)=10^3 \sin\pi(3\times10^6 x+9\times10^8 t+0.5)$ in SI units find the following quantities:

a) speed, b) wavelength, c) frequency, d) period, e) amplitude,

f) phase, g) initial phase, h) phase at $t=10$ s, i) phase velocity,

j) compare the speed and the phase velocity.

k) what is the direction of the motion of the wave. Does phse velocity indicate the direction of motion properly?

k) what is magnitude of the wave at $x=0$ when $t=0,\tau/4,\tau/2,3\tau/4,\tau$?

l) plot the profile of the wave at $t=0$ with initial pahse equal to $0,\ 0.5\pi,\ \pi,\ 2\pi.$

Complex numbers representation

More often wavefunctions are expressed in complex exponentials since complex exponentials can simplify the trigonometric expressions.

$z = x + jy$ or $z = x + iy$; i or $j = \sqrt{-1}$ and both x and y are real numbers.

$|z| = \sqrt{x^2 + y^2}; \quad \tan\phi = \frac{y}{x}$

We can also write $z = Ae^{i\phi}$

Euler's formula: $e^{i\phi} = \cos\phi + i\sin\phi$

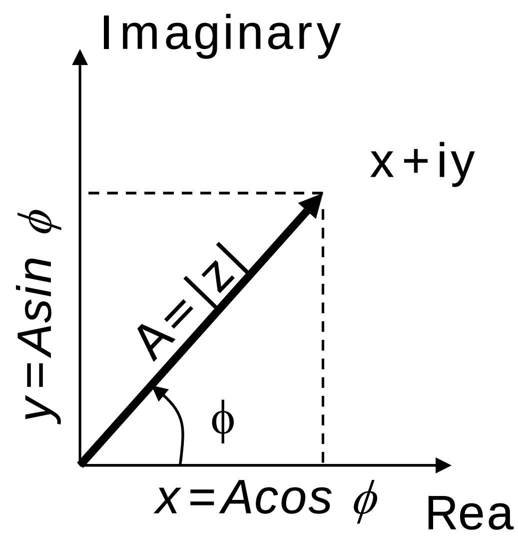
$z = A(\cos\phi + i\sin\phi)$ where

$x = A\cos\phi$ and $y = A\sin\phi$

$A = \sqrt{x^2 + y^2}$ is the magnitude or amplitude

$\phi = \tan^{-1}\left(\frac{y}{x}\right)$ is the phase

$z^* = x - iy$ is the complex conjugate of z.



Argand diagram

Exercise

1.11) Show that: $a) zz^* = A^2$; $b) |A| = \sqrt{zz^*}$, $c) \mathrm{Re}(z) = \frac{1}{2}(z + z^*)$;

$d) \cos\phi = \dfrac{e^{i\phi} + e^{-i\phi}}{2}$; $e) \sin\phi = \dfrac{e^{i\phi} - e^{-i\phi}}{2i}$; $f)\ z_1 z_2 = A_1 A_2 e^{i(\phi_1 + \phi_2)}$ if $z_1 = A_1 e^{i\phi_1}$ and $z_2 = A_2 e^{i\phi_2}$; $g)\ \psi = Ae^{i\phi}$ is unchanged if the phase is increased or decreased by 2π; h) multiplying a complex wavefunction by $\pm i$ is equivalent to shifting its phase by $\pm\pi/2$ (Hint: first show $\pm i = e^{\pm i\pi/2}$); i) Show that if two harmonic waves (cosine) with same amplitude, speed, and frequency (one forward going the other backward) with π difference in phase overlap, the result is: $\psi(y,t) = -2A \sin ky \sin \omega t$ Plot this result. Have you seen it before? Interprete your results.

Harmonic plane wave I

Plane waves are very important in optics.

For the plane waves the wavefront is planar.

All the points on a wavefront share a contant phase.

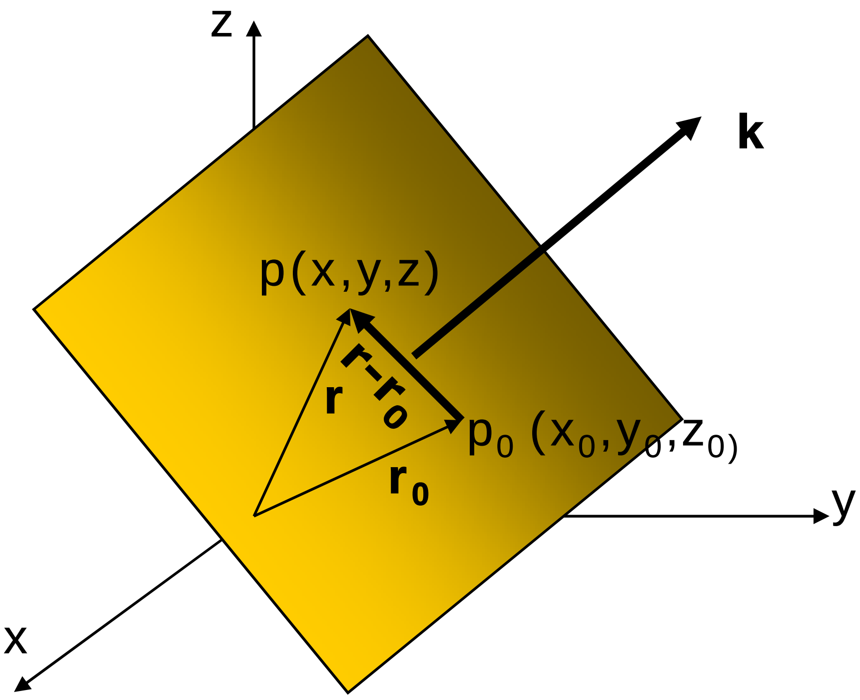
Write equation of a plane that is passing through $p_0(x_0,\ y_0,\ z_0)$ and is perpendicular to a given direction, $\mathbf{k}$ (direction of propagation).

Consider an arbitrary point p on the plane. We must have:

$(\mathbf{r} - \mathbf{r_0})\bullet\mathbf{k} = \mathbf{0}$.

Thus $\mathbf{k}\bullet\mathbf{r} = \mathbf{k}\bullet\mathbf{r_0} = constnt$

is equation of such a plane perpendicular to $\mathbf{k}$ (2.7 Hecht).



Harmonic plane wave II

$\mathbf{k}\bullet\mathbf{r} = \mathbf{k}\bullet\mathbf{r_0} = constnt$ is equation of a plane perpendicular to $\mathbf{k}$ and passing through p_0.

$\psi(\mathbf{r}) = A\sin(\mathbf{k}\bullet\mathbf{r})$ is a function defined on a family of planes all perpendicular to $\mathbf{k}$. Phase of this function stays the same as long as its value is $\mathbf{k}\bullet\mathbf{r} = \mathbf{k}\bullet\mathbf{r_0} =$ constant

On each of these planes $\psi(\mathbf{r})$ is constant

but from plane to plane $\psi(\mathbf{r})$ varies sinusoidally.

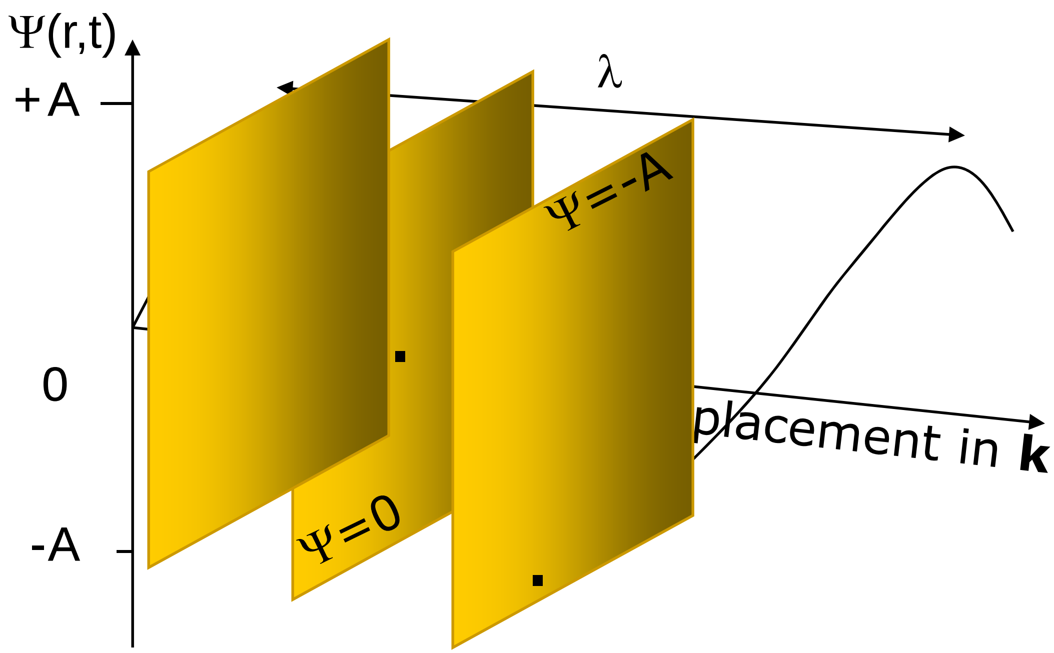
The progressive wave equation is then:

$\psi(\mathbf{r},t) = A\sin(\mathbf{k}\bullet\mathbf{r} \mp \omega t)$ or

$\psi(\mathbf{r},t) = Ae^{i(\mathbf{k}.\mathbf{r}\mp\omega t)} = A\cos(\mathbf{k}\bullet\mathbf{r} \mp \omega t) + iA\sin(\mathbf{k}\bullet\mathbf{r} \mp \omega t)$

Amplitude of a plane wave stays constant as it propagates

Value of the plane wavefunction along the propagation direction



Harmonic spherical wave

By solving the differential spherical wave equation we can arrive at harmonic spherical wave function (2.9 on Hecht):

$$\psi(r,t) = \frac{A}{r}\sin k(r \mp Vt); \qquad \psi(r,t) = \frac{A}{r}e^{ik(r \mp Vt)}$$

that represents cluster of concentric shperes at any instance.

On each sphere r is constant so $\psi(r,t)$ is constant.

Here A, the source strength is a constant.

$\frac{A}{r}$ is amplitude and it varies inversly with the distance from the source to conserve the energy.

Wavefront

- Wavefront: a surface over which the phase of a wave is constant.
- For plane waves the wavefronts are planar surfaces for which **k.r=**constant. For spherical waves the wavefronts are concentric spheres centered at origin of the wave.
- Homogeneous waves: the wavefunction is constant over the wavefront, i.e. amplitude is constant.
- Inhomogeneous waves: the wavefunction is not constant over the wavefront.
- Example of an inhomogeneous wave function?

Exercise

1.12) Show that $\psi(x, y, z) = Ae^{ik(\alpha x+\beta y+\gamma z\mp Vt)}$ is a solution to the 3D wave equation. Here α, β, γ are the direction cosines.

(Hint: $\alpha = k_x / k,\ \beta = k_y / k,\ \gamma = k_z / k$ And remember $\alpha^2 + \beta^2 + \gamma^2 = 1$)

1.13) Show that $\psi(r, t) = \dfrac{f(r - Vt)}{r}$ is the solution to the 3D wave equation which corresponds to a spherical disturbance centered at the origin and moving outward with speed of V. Here $f(r - Vt)$ is an arbitrary twice differentiable function (Hint: Laplasian is not the same in all coordinate systems).

Electromagnetic waves

- The harmonic wave equation can represent any type of disturbance with sinusoidal behavior.
- Physical significance of the disturbance is different for different systems.
- Maxwell showed that light is composed of electric and magnetic fields oscillating perpendicular to each other and propagating in the direction k, perpendicular to the plane of oscillations.
- For light waves the disturbance is the magnitude of varying electric or magnetic fields that are described with the following harmonic wavefunctions:

$$\mathbf{E} = \mathbf{E_0} e^{i(\mathbf{k}.\mathbf{r}-\omega t)}, \quad \mathbf{B} = \mathbf{B_0} e^{i(\mathbf{k}.\mathbf{r}-\omega t)}$$

Electric and magnetic fields

Electric fields are generated by electric charges and time-verying magnetic fields.

Magnetic fields are generated by electric currents (charge in motion) and time-varying electric fields.

This intedependence of the $\mathbf{E}$ and $\mathbf{B}$ is a key point in description of the light.

Lorentz force: an electric charge q, moving with velocity of $\mathbf{v}$ in an area that contains both $\mathbf{E}$ and $\mathbf{B}$ fileds, feels forces due to existance of both fields.

$$\mathbf{F} = \mathrm{q} \cdot (\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

Before Maxwell

Faraday's induction law: a time-varying magnetic field will have an electric field associated with it. $\oint_C \mathbf{E} \cdot d\vec{l} = - \iint_{\mathbf{A}} \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{s}$

Gauss's law-electric: $\oiint_{\mathrm{A}} \mathbf{E} \cdot d\mathbf{s} = \frac{1}{\varepsilon_0} \iiint_V \rho dV,$ when there are no sources or sinks of the electric field within the region encompassed by a closed surface, the net flux through the surface equals to zero.

Gauss's law-magnetic: $\Phi_{\mathrm{M}} = \oiint_{\mathrm{A}} \mathbf{B} \cdot d\mathbf{s} = 0,$ there is no magnetic monopole

Ampere's circuital law: $\oint_C \mathbf{B} \cdot d\vec{l} = \mu \iint_{\mathbf{A}} \left(\mathbf{J} + \varepsilon \frac{\partial \mathbf{E}}{\partial t} \right) \cdot d\mathbf{s},$ a time-varying $\mathbf{E}$-field or charges in motion (electric current) will generate a $\mathbf{B}$-field

Maxwell equations; integral form

Behavior of electric and magnetic fields in a medium with electric permitivity ε, and magnetic permeability μ, in presence of free charges ρ, and current density $\mathbf{J}$, is explained by four integral equations known as Maxwell equations.

$$\oint_C \mathbf{E} \cdot d\vec{l} = -\iint_{\mathbf{A}} \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{s}$$

$$\oint_C \frac{\mathbf{B}}{\mu} \cdot d\vec{l} = \iint_{\mathbf{A}} \left(\mathbf{J} + \varepsilon \frac{\partial \mathbf{E}}{\partial t} \right) \cdot d\mathbf{s}$$

$$\oiint_{\mathrm{A}} \mathbf{B} \cdot d\mathbf{s} = 0$$

$$\oiint_{\mathrm{A}} \varepsilon \mathbf{E} \cdot d\mathbf{s} = \iiint_V \rho dV$$

Compact notation: good-looking, and cool equations

Maxwell equations; differential form

$$\oint_C \mathbf{E}\cdot d\vec{l} = -\iint_{\mathbf{A}} \frac{\partial \mathbf{B}}{\partial t}\cdot d\mathbf{s} \qquad \rightarrow \nabla\times\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\oint_C \frac{\mathbf{B}}{\mu}\cdot d\vec{l} = \iint_{\mathbf{A}} \left(\mathbf{J} + \varepsilon\frac{\partial \mathbf{E}}{\partial t}\right)\cdot d\mathbf{s} \qquad \rightarrow \nabla\times\mathbf{B} = \mu\left(\mathbf{J} + \varepsilon\frac{\partial \mathbf{E}}{\partial t}\right)$$

$$\oiint_{\mathrm{A}} \mathbf{B}\cdot d\mathbf{s} = 0 \qquad \rightarrow \nabla\cdot\mathbf{B} = 0$$

$$\oiint_{\mathrm{A}} \varepsilon\mathbf{E}\cdot d\mathbf{s} = \iiint_V \rho\, dV \qquad \rightarrow \nabla\cdot\mathbf{E} = \frac{\rho}{\varepsilon}$$

We used the following theorems: Stoke's $\oint_C \mathbf{E}\cdot d\vec{l} = \iint_{\mathbf{A}} \nabla\times\mathbf{E}\cdot d\mathbf{s}$

Gauss's $\oiint_{\mathrm{A}} \mathbf{B}\cdot d\mathbf{s} = \iiint_V \nabla\cdot\mathbf{B}\, dV$

Constitutive relations

$\mathbf{H} =$ Magnetic field; $\mathbf{B} =$ Magnetic induction;

$\mathbf{E} =$ Electric field; $\mathbf{D} =$ Electric displacement; $\mathbf{J} =$ Current density;

Constituitive relations are: $\mathbf{D} = \mathbf{D}(\mathbf{E},\mathbf{B})$; $\mathbf{H} = \mathbf{H}(\mathbf{E},\mathbf{B})$; $\mathbf{J} = \mathbf{J}(\mathbf{E},\mathbf{B})$

These relations may be nonliner or depend on the past (hysteresis).

Linear response: the applied fields are small so they induce electric and magnetic polarizations proportional to the magnitude of the applied field (ferroelectric and ferromagnetic material are exceptions)

$$\mathbf{D}_\alpha = \sum_\beta \varepsilon_{\alpha\beta} E_\beta; \quad \mathbf{H}_\alpha = \sum_\beta \mu^{-1}_{\alpha\beta} B_{\alpha\beta}$$

$\varepsilon_{\alpha\beta}$ is electric permitivity or dielectric tensor

$\mu^{-1}_{\alpha\beta}$ is inverse magnetic permitivity tensor

For material isotropic in space both ε and μ are diagonal and all elements are equal then: $\mathbf{D}=\varepsilon\mathbf{E}$ and $\mathbf{H}=\mu^{-1}\mathbf{B}$

At high enough fields every material is nonlinear (nonlinear optics)

$$D_\alpha = \sum_\beta \varepsilon^{(1)}_{\alpha\beta} E_\beta + \sum_{\beta,\gamma} \varepsilon^{(2)}_{\alpha\beta\gamma} E_\beta E_\gamma + \ldots.$$

for most of optical material $\mu = 1$

Electric fields in medium

1) $\mathbf{P}$ polarization vector: $\underline{\mathbf{P} = \varepsilon_0 \chi \mathbf{E}}$ electric dipole moment per unit volume.

2) $\mathbf{D}$ displacement field: $\underline{\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P}}$ electric field within the material

3) $\mathbf{E}$ internal electric field: $\mathbf{E} = \dfrac{\mathbf{D}}{\varepsilon_{\mathbf{0}}} - \dfrac{\mathbf{P}}{\varepsilon_0}$ and we have $\underline{\mathbf{D} = \varepsilon(\mathbf{E})\ \mathbf{E}}$

$\mathbf{D}$ and $\mathbf{E}$ lines begin and end on free charges or polarization cahrges.

In absence of free charge field lines close on temselves

$(\nabla \cdot \mathbf{E} = \nabla \cdot \mathbf{D} = 0)$.

<u>For homogeneous, linear, isotropic dielectrics</u> $\mathbf{P}$ and $\mathbf{E}$ are in the same direction so $\mathbf{D} = \varepsilon \mathbf{E},$ where $\varepsilon = \varepsilon_0(1+\chi)$, and $\mathrm{K}_e = \varepsilon / \varepsilon_0 = 1 + \chi$ is the dielectric constant or function.

4) Ohm's law: electric field intensity determines the flow of cahrge in a conductor $\underline{\mathbf{J} = \sigma \mathbf{E}},$ true for conductors at constant temperature.

Constitutive relations: magnetic fields in a medium

1) $\mathbf{M}$ magnetic polarization vector: $\mathbf{M} = \mathrm{K}_m \mathbf{H}$.

2) $\mathbf{H}$ magnetic field intensity: $\underline{\mathbf{H} = \mu_{\mathbf{0}}{}^{-1}\mathbf{B} - \mathbf{M}}$,

<u>For homogeneous, linear (nonferromagnetic), isotropic</u>

medium $\mathbf{B}$ and $\mathbf{H}$ are parallel and proportional,

$\underline{\mathbf{H} = \mu^{-1}\mathbf{B}}$ and $\mu = \mu_0(1 + \mathrm{K}_m)$ and $\mu_r = \dfrac{\mu}{\mu_0}$

Maxwell equations using the constitutive relations

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad \rightarrow \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon} \qquad \rightarrow \nabla \cdot \mathbf{D} = \rho$$

$$\nabla \times \mathbf{B} = \mu\left(\mathbf{J} + \varepsilon \frac{\partial \mathbf{E}}{\partial t}\right) \rightarrow \nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$

$$\nabla \cdot \mathbf{B} = 0 \qquad \rightarrow \nabla \cdot \mathbf{B} = 0$$

The wave equation for the E and M components of the EM waves

Maxwell equations in nonconducting $(\rho = 0,\ \mathbf{J} = 0)$ vacuum

$$\varepsilon = \varepsilon_0,\ \mu = \mu_0, \qquad (1)\ \ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \qquad (2)\ \ \nabla \times \mathbf{B} = \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

$$(3)\ \ \nabla \cdot \mathbf{B} = 0, \qquad (4)\ \ \nabla \cdot \mathbf{E} = 0$$

Take curl of (1) and use (2) to eliminate $\mathbf{B}$, $\nabla \times \nabla \times \mathbf{E} = -\frac{\partial}{\partial t}(\nabla \times \mathbf{B})$,

$\nabla \times \nabla \times \mathbf{E} = -\mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2},\ \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} = -\mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2}$, using (4) we get

differential wave equation: $\boxed{\nabla^2 \mathbf{E} = \mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} = \frac{1}{\mathbf{V}^2} \frac{\partial^2 \mathbf{E}}{\partial t^2},}$ with $|\mathbf{V}| = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$

We can also show: $\boxed{\nabla^2 \mathbf{B} = \mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{B}}{\partial t^2} = \frac{1}{\mathbf{V}^2} \frac{\partial^2 \mathbf{B}}{\partial t^2},}$

Exercise

1.14) a) Given a harmonic plane electromagnetic wave whose $\mathbf{E}-$field is: $E(x,t)=E_{0y}\sin\left[\omega\left(t-\frac{x}{c}\right)+\varepsilon\right]$

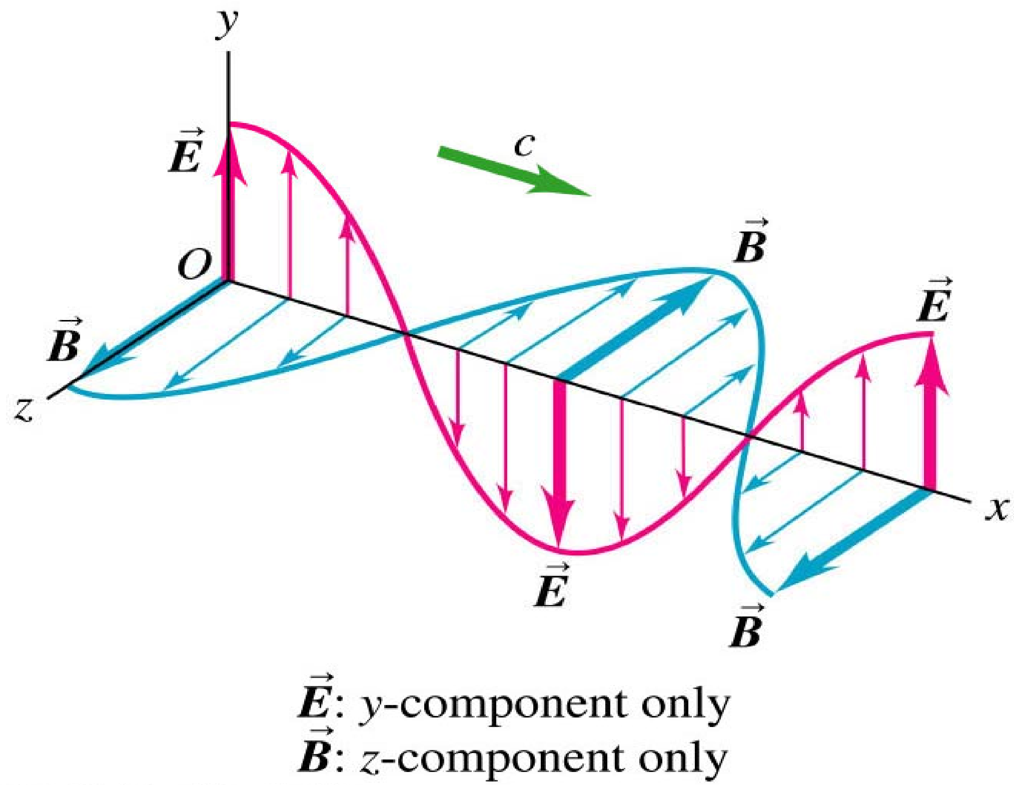
determine the corresponding $\mathbf{B}$ field, direction of both $\mathbf{E}$ and $\mathbf{B}$ fields. Make a sketch of the wave.

b) What is the direction of propagation of this wave?

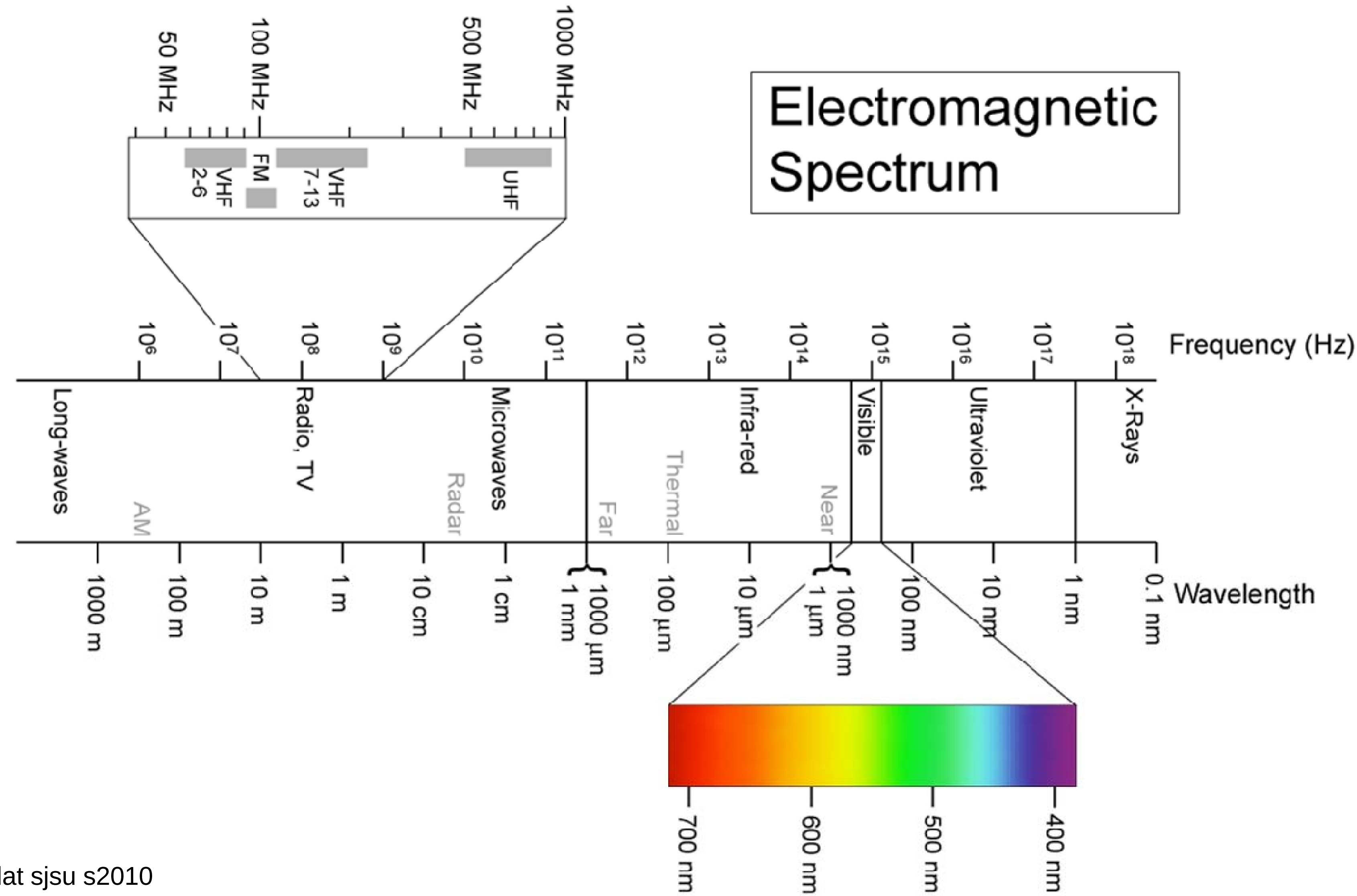
c) Prove that for the above harmonic wave, direction of the propagation vector is along the $+x$ direction.

Hint: propagation of an electromagnetic wave is along the cross product of $\mathbf{E}$ and $\mathbf{B}$ or $\mathbf{E}\times\mathbf{B}$.

Exercise RV1-14



The electromagnetic spectrum



The index of refraction

Velocity of light based on Maxwell's theoretical treatment in vacuum is

$c = \dfrac{1}{\sqrt{\varepsilon_0 \mu_0}}$ and in medium is $V = \dfrac{1}{\sqrt{\varepsilon \mu}}$.

Absolute index of refraction defined as: $n = \dfrac{c}{V} = \sqrt{\dfrac{\varepsilon \mu}{\varepsilon_0 \mu_0}} = \sqrt{K_e \mu_r}$.

The $\mathbf{E}$ field of an EM wave polarizes the medium and the displacement field $\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P} = \varepsilon \mathbf{E}$ changes. The result is change in ε and consequently in n and $V,$ speed of light in the medium.

$\varepsilon(\omega)$ and $n(\omega)$ are functions of the frequency of the EM waves.

Usually $\mu \approx \mu_0$ so $n = \sqrt{\dfrac{\varepsilon}{\varepsilon_0}} = \sqrt{K_e}$; K_e is the complex dielectric function.

Physical meaning of the index of refraction I

Consider a general case where n is a complex number $n = n' + in''$. Consider the E component of a plane wave, $\mathbf{E} = \mathbf{E}_0 e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}$ where $\mathbf{k},$ the propagation vector in medium is complex. $\mathbf{k}_0$ is propagation vector in vacuum.

With $V = \dfrac{\omega}{|\mathbf{k}|}$, the phase velocity, and $c = \dfrac{\omega}{|\mathbf{k}_0|} = nV$

$\rightarrow |\mathbf{k}| = n|\mathbf{k}_0| = \dfrac{\omega}{V} = \dfrac{\omega n}{c}$ for one-dimensional case

$$E = E_{0y} e^{i(kx - \omega t)} = E_{0y} e^{i(\frac{n\omega}{c}x - \omega t)} = E_{0y} e^{i(\frac{(n' + in'')\omega}{c}x - \omega t)} = E_{0y} e^{-\frac{n''\omega x}{c}} e^{i\omega(\frac{n' x}{c} - t)}$$

Physical meaning of the index of refraction II

$$E = E_{0y} e^{-\frac{n''\omega x}{c}} e^{i\omega(\frac{n'x}{c} - t)}$$

$e^{-\frac{n''\omega x}{c}}$ is a real term and decays exponentially as wave propagates.

$e^{i\omega(\frac{n'x}{c} - t)}$ has a harmonic wave form and propagates without loss

This suggests that

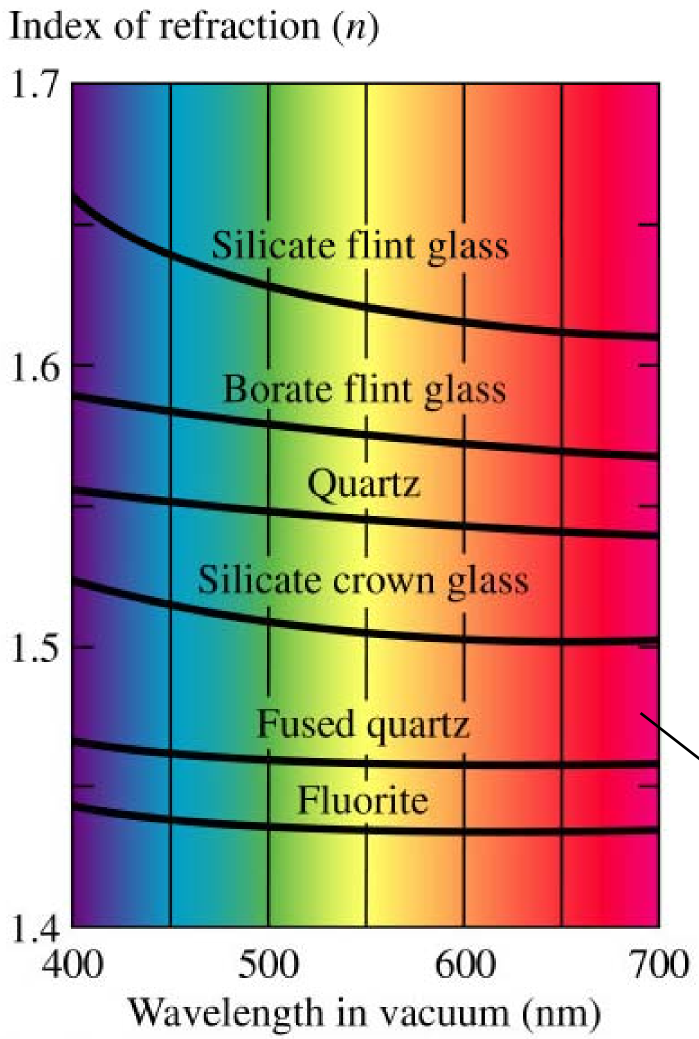
n'', the imaginary part of n is associated with absorption.

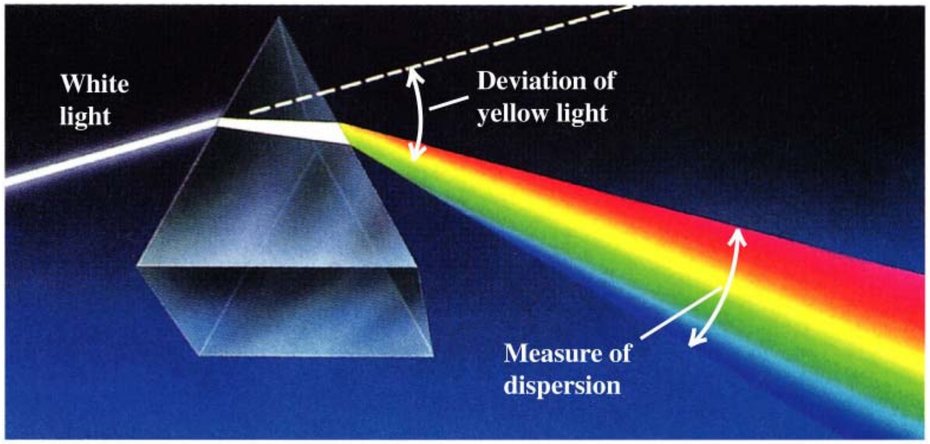
n', the real part of n is associated with propagation.

In absence of n'', the case in many dielectrics in the visible part of the EM spectrum, $n = n' = c/V$ is simply the ratio of the speeds in vacuum and in the medium.

$n = n' = c/V$ is also called absolute index of refraction.

Frequency dependence of the index of refraction





The wavelength dependence of n is stronger at short wavelengths or high frequencies.
For most dielectrics the imaginary part of the n is negligible in the visible band

Exercise

1.15) Suppose a light wave propagates from point A to point B. We introduce a glass plate $(n = 1.50)$ of thickness 1m into its path.

a) How much phase of the wave will be altered at point B if $\lambda_0 = 500\ nm$? Assume $n_{air} = 1.00$.

b) What is the phase velocity of the wave in the glass?

Energy and momentum I

Physical manifestation of electromagnetic waves is their enegy and momentum.

Energy density u, is radiant energy per unit volume

In order to calculate the energy content of the EM waves we start from

$$\nabla\cdot(\mathbf{E}\times\mathbf{H})=\mathbf{H}\cdot(\nabla\times\mathbf{E})-\mathbf{E}\cdot(\nabla\times\mathbf{H})$$

Using $\nabla\times\mathbf{E}=-\frac{\partial\mathbf{B}}{\partial t}$ and $\nabla\times\mathbf{H}=\mathbf{J}+\frac{\partial\mathbf{D}}{\partial t}$ and $\mathbf{J}=\sigma\mathbf{E}$

$$\nabla\cdot(\mathbf{E}\times\mathbf{H})=\mathbf{H}\cdot(-\frac{\partial\mathbf{B}}{\partial t})-\mathbf{E}\cdot(\frac{\partial\mathbf{D}}{\partial t})-\mathbf{E.J}=\mathbf{H}\cdot(-\frac{\partial\mathbf{B}}{\partial t})-\mathbf{E}\cdot(\frac{\partial\mathbf{D}}{\partial t})-\sigma\mathbf{E.E}$$

$$\mathbf{H}\cdot\frac{\partial\mathbf{B}}{\partial t}=\mathbf{H}\cdot\frac{\partial(\mu\mathbf{H})}{\partial t}=\frac{1}{2}\frac{\partial(\mu\mathbf{H.H})}{\partial t}=\frac{\partial}{\partial t}\left(\frac{1}{2}\mu H^2\right)$$

$$\mathbf{E}\cdot\frac{\partial\mathbf{D}}{\partial t}=\mathbf{E}\cdot\frac{\partial(\varepsilon\mathbf{E})}{\partial t}=\frac{1}{2}\frac{\partial(\varepsilon\mathbf{E.E})}{\partial t}=\frac{\partial}{\partial t}\left(\frac{1}{2}\varepsilon E^2\right)$$

$$\nabla\cdot(\mathbf{E}\times\mathbf{H})=-\frac{\partial}{\partial t}\left(\frac{1}{2}\varepsilon E^2+\frac{1}{2}\mu H^2\right)-\sigma E^2$$

Energy and momentum II

$$\nabla\cdot(\mathbf{E}\times\mathbf{H}) = -\frac{\partial}{\partial t}\left(\frac{1}{2}\varepsilon E^2 + \frac{1}{2}\mu H^2\right) - \sigma E^2$$

Integrating both sides over a volume V, and applying the

divergence theorem $\int_V \nabla\cdot(\mathbf{E}\times\mathbf{H})\,dV = \oint_S \mathbf{E}\times\mathbf{H}\cdot d\mathbf{s}$

$$\oint_S \mathbf{E}\times\mathbf{H}\cdot d\mathbf{s} = -\frac{\partial}{\partial t}\int_V \left(\frac{1}{2}\varepsilon E^2 + \frac{1}{2}\mu H^2\right) dV - \int_V \sigma E^2 dV$$

This is equal to power rate leaving the volume from its surface.

First two terms of RHS: time-rate of change of energy stored in electric and magnetic fields.

Third term of RHS: ohmic power dissipated in the volume as a result of conduction current density $\sigma\mathbf{E}$ in presence of $\mathbf{E}$.

Thus the $\mathbf{E}\times\mathbf{H}$ is a vector representing power flow per unit volume.

Poynting's Theorem

$\mathbf{S} = \mathbf{P} = \mathbf{E} \times \mathbf{H} = \frac{1}{\mu}\ \mathbf{E} \times \mathbf{B}\ \ (W/m^2)$ is known as Poynting vector.

$|\mathbf{S}| =$ power density crossing a surface whose normal is parallel to $\mathbf{S}$.

Poynting's theorem: surface integral of $\mathbf{P}$ or $\mathbf{S}$ over a closed

surface $S,$ equals the power leaving the enclosed volume V.

$$-\oint_S \mathbf{S}.d\mathbf{s} = \frac{\delta}{\delta t}\int_V (u_e + u_m)dV + \int_V P_\sigma dV \quad \text{where}$$

$$u_e = \frac{1}{2}\varepsilon E^2 = \frac{1}{2}\varepsilon \mathbf{E}\cdot\mathbf{E}^* = \frac{1}{2\varepsilon}\mathbf{D}\cdot\mathbf{D}^* = \text{Electric energy density}$$

$$u_e = \frac{1}{2}\mu H^2 = \frac{1}{2}\mu \mathbf{H}\cdot\mathbf{H}^* = \frac{1}{2\mu}\mathbf{B}\cdot\mathbf{B}^* = \text{Magnetic energy density}$$

$$P_\sigma = \sigma E^2 = \sigma \mathbf{E}\cdot\mathbf{E}^* = \ \text{Ohmic power density}$$

Exercise

1.16) a) Prove that for a harmonic plane electromagnetic wave with the following $\mathbf{E}$ field component traveling through an insulating isotropic medium, $|\mathbf{E}| = V|\mathbf{B}|$. In vacuum $|\mathbf{E}| = c|\mathbf{B}|$.

$\mathbf{E} = \mathbf{E}_0 \cos(\mathbf{k} \cdot \mathbf{r} - \omega t)$

b) Calculate the magnitude of the poynting vector

c) Prove that the energy content of this plane wave, in unit volume, due to electric and magnetic fields is equal.

d) Calculate the total energy per unit volume.

Irradiance

Irradiance I, or the amount of light: average energy over unit area per unit time. I is independent of detector area A, and duration of measurement.

Time averaged value of the Poynting vector $\mathbf{S}$, or irradiance is:

$$I \equiv \left\langle \mathbf{S} \right\rangle_T = \frac{1}{T} \int_{t-\frac{T}{2}}^{t+\frac{T}{2}} (\mathbf{E} \times \mathbf{H}) dt$$

Note that the averaging time $T \gg \tau$, has to be much greater than the period.

Irradince is proportional to the square of the amplitude of the $\mathbf{E}$ field.

For linear, homogeneous, isotropic dielectric

$I = \varepsilon V \left\langle E^2 \right\rangle_T$ In vacuum: $I = \dfrac{c}{\mu_0} \left\langle B^2 \right\rangle_T = \varepsilon_0 c \left\langle E^2 \right\rangle_T$

Exercise

1.17) a) Calculate time average of the harmonic plane wave

$E = E_0 e^{i(kx-\omega t)}$ over a time of T (T is not temporal period of the wave).

b) Calculate the $\langle \cos(kx-\omega t)\rangle_T$, $\langle \sin(kx-\omega t)\rangle_T$, $\langle \cos(kx-\omega t)\rangle_T^2$,

$\langle \sin(kx-\omega t)\rangle_T^2$. Use the notation $\sin c\ x = \dfrac{\sin x}{x}$.

c) Use the results of parts a and b to show that for plane waves

in vacuum $I \equiv \langle \mathbf{S} \rangle_T = \dfrac{c\varepsilon_0}{2} E_0^2$

d) Calculate the optical flux density for the plane EM wave with the following electrical (also called optical) field moving in vacuum

$$E_x = E_y = 0, \quad E_z = 100 \sin\left[8\pi \times 10^{14}\left(t - \frac{x}{3\times 10^8}\right)\right]. \quad (Ans:\ 13.3\ W/m^3)$$

Exercise

1.18) A 60 watts monochromatic point source radiating equally in all directions in vacuum is being monitored at a distance of $2.0\ m$. Using the the fact that $\mu_0 = 4\pi \times 10^{-7}\ Ns^2C^{-2}$, determine the amplitude of the $\mathbf{E}-$field at the detector. (Ans: $E_0 = 30\ V/m$)