Chapter 18
Matrix Methods in Paraxial Optics

Lecture Notes for Modern Optics based on Pedrotti & Pedrotti & Pedrotti
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Matrix methods in paraxial optics

- Describing a single thick lens in terms of its cardinal points.
- Describing a single optical element with a 2x2 matrix.
- Analysis of train of optical elements by multiplication of 2x2 matrices describing each element.
- Computer ray-tracing methods, a more systematic approach
Cardinal points and cardinal planes

We define six cardinal points on the axis of a thick lens from which its imaging properties can be deduced.

Planes normal to the axis at the cardinal points are called cardinal planes.

Cardinal points and planes include

First and second set of focal points and focal planes.

First and second principal points and principal planes. The rays determining the focal points change direction at their intersection with the principal planes.
Cardinal points and cardinal planes

First and second nodal points and nodal planes. Nodal points of a thick lens or any optical system permit correction to the ray that aims the center of the lens. Any ray that aims the first nodal point emerges from the second nodal point undeviated but slightly displaced.
Cardinal points and cardinal planes

All the distances that are directed to the left are negative (-) and directed to the right are positive (+) by the sign convention. Notice that focal distances are not measured from the vertices.
Basic equations for the thick lens

\[
\frac{1}{f_1} = \frac{n_L - n'}{nR_2} - \frac{n_L - n}{nR_1} - \frac{(n_L - n)(n_L - n')}{nn_L} R_1 R_2
\]

\[
f_2 = -\frac{n'}{n} f_1 \quad \text{for } n = n' \text{ then } f_2 = -f_1
\]

Location of the principal planes:

\[
r = \frac{n_L - n'}{n L R_2} f_1 t; \quad s = \frac{n_L - n}{n L R_1} f_2 t
\]

The positions of the nodal points:

\[
v = \left(1 - \frac{n'}{n} + \frac{n_L - n'}{n L R_2} t\right) f_1; \quad w = \left(1 - \frac{n'}{n} + \frac{n_L - n}{n L R_1} t\right) f_2
\]

Image and object distances and lateral magnification:

\[
-\frac{f_1}{s_o} + \frac{f_2}{s_i} = 1 \quad \text{and} \quad m = -\frac{ns_i}{n's_o}
\]

The sign convention is as usual (real + and virtual −) as long as the distances are measured relative to their corresponding principal planes.

For an ordinary thin lens in air: \(n = n' = 1\) and \(r = v, \ s = w\) we arrive at the usual thin lens equations:

\[
\frac{1}{s_o} + \frac{1}{s_i} = \frac{1}{f} \quad \text{and} \quad m = -\frac{s_i}{s_o} \quad \text{and} \quad f = f_2 = -f_1
\]
The matrix methods in paraxial optics

For optical systems with many elements we use a systematic approach called matrix method.
We follow two parameters for each ray as it progresses through the optical system.
A ray is defined by its height and its direction (the angle it makes with the optical axis).
We can express \( y_7 \) and \( \alpha_7 \) in terms of \( y_1 \) and \( \alpha_1 \) multiplied by the transfer matrix of the system.

![Diagram of optical system with rays and transfer matrices.](image)
The translational matrix

Consider simple translation of a ray in a homogeneous medium.

Translation from point 0 to 1 with paraxial approximation:

\[ \alpha_1 = \alpha_0 \text{ and } y_1 = y_0 + L \tan \alpha_0 = y_0 + L \alpha_0 \]

We rewrite the equations:

\[
\begin{align*}
    y_1 &= (1) y_0 + (L) \alpha_0 \\
    \alpha_1 &= (0) y_0 + (1) \alpha_0
\end{align*}
\]

\[
\begin{bmatrix}
    y_1 \\
    \alpha_1
\end{bmatrix}
= 
\begin{bmatrix}
    1 & L \\
    0 & 1
\end{bmatrix}
\begin{bmatrix}
    y_0 \\
    \alpha_0
\end{bmatrix}
\]

Ray-transfer matrix for translation
**Refraction matrix**

Consider refraction of a ray at a spherical interface (paraxial approximation):

Ray coordinates before refraction \((y, \alpha)\) and ray coordinates after refraction \((y', \alpha')\)

\[\alpha' = \theta' - \phi = \theta' - \frac{y}{R}\]

and

\[\alpha = \theta - \phi = \theta - \frac{y}{R}\]

Paraxial form of Snell's law: \(n\theta = n'\theta'\)

\[\alpha' = \left(\frac{n}{n'}\right) \theta - \frac{y}{R} = \left(\frac{n}{n'}\right) \left(\alpha + \frac{y}{R}\right) - \frac{y}{R}\]

\[\alpha' = \left(\frac{1}{R}\right) \left(\frac{n}{n'} - 1\right) y + \left(\frac{n}{n'}\right) \alpha\]

The approximate linear equations:

\[y' = (1) y + (0) \alpha\]

\[\alpha' = \left[\left(\frac{1}{R}\right) \left(\frac{n}{n'} - 1\right)\right] y + \left(\frac{n}{n'}\right) \alpha\]

\[\begin{bmatrix} y' \\ \alpha' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{1}{R} \left(\frac{n}{n'-1}\right) & \frac{n}{n'} \end{bmatrix} \begin{bmatrix} y \\ \alpha \end{bmatrix}\]

Ray-transfer matrix for refraction

If \(R \to \infty\) we have transfer matrix for refraction by plane interface:

\[\begin{bmatrix} y' \\ \alpha' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{n}{n'} \end{bmatrix} \begin{bmatrix} y \\ \alpha \end{bmatrix}\]
**The reflection matrix**

Consider refraction of a ray at a spherical interface (paraxial approximation):
Ray coordinates before refraction \((y, \alpha)\) and ray coordinates after refraction \((y', \alpha')\)

\[\alpha' = \theta' - \phi = \theta' - \frac{y}{R} \quad \text{and} \quad \alpha = \theta - \phi = \theta - \frac{y}{R}\]

Goal: connect \((y', \alpha')\) to \((y, \alpha)\) by a ray transfer matrix for reflection by a concave mirror

Sign convention for the angles: (+) pointing upward and (−) pointing downward

\[\alpha = \theta + \phi = \theta + \frac{y}{-R} \quad \text{and} \quad \alpha' = \theta' - \phi = \theta' - \frac{y}{-R}\]

To eliminate \(\theta\) and \(\theta'\) we use \(\theta = \theta'\) we get

\[\alpha' = \theta + \frac{y}{R} = \alpha + \frac{2y}{R}\]

The desired equations become:

\[y' = (1) y + (0) \alpha\]

\[\alpha' = \left(\frac{2}{R}\right) y + (1) \alpha\]

\[
\begin{bmatrix}
\alpha' \\
y'
\end{bmatrix} =
\begin{bmatrix}
\frac{2}{R} & 1 \\
1 & 0
\end{bmatrix}
\begin{bmatrix}
y \\
\alpha
\end{bmatrix}
\]

Ray-transfer matrix for reflection
The thick lens and thin lens matrices
The general Ray-transfer matrix

Goal: construct a matrix that represents a thick lens with two different material on each side of it. In traversing the lens the ray undergoes two refractions and one translation for which we have derived the matrices. The radii of curvature are (+) in this example. The symbolic equations are:

\[
\begin{bmatrix}
y_1 \\
\alpha_1
\end{bmatrix} = M_1 \begin{bmatrix}
y_0 \\
\alpha_0
\end{bmatrix} \quad \text{for the first reflection}
\]

\[
\begin{bmatrix}
y_2 \\
\alpha_2
\end{bmatrix} = M_2 \begin{bmatrix}
y_1 \\
\alpha_1
\end{bmatrix} \quad \text{for the translation}
\]

\[
\begin{bmatrix}
y_3 \\
\alpha_3
\end{bmatrix} = M_3 \begin{bmatrix}
y_2 \\
\alpha_2
\end{bmatrix} \quad \text{for the second reflection}
\]

\[
\begin{bmatrix}
y_3 \\
\alpha_3
\end{bmatrix} = M_3 M_2 M_1 \begin{bmatrix}
y_0 \\
\alpha_0
\end{bmatrix}
\]

The individual matrix operates on the ray in the same order in which the optical acts influence the ray. No commutative property for multiplication of matrices. Only associative property holds.

\[
M = M_3 M_2 M_1 = (M_3 M_2) M_1 = M_3 (M_2 M_1) \neq M_2 M_3 M_1
\]

Generalizing the matrix relationship for any number of translating, reflecting, refracting surfaces:

\[
\begin{bmatrix}
y_f \\
\alpha_f
\end{bmatrix} = M_N M_{N-1} \cdots M_2 M_1 \begin{bmatrix}
y_0 \\
\alpha_0
\end{bmatrix}
\]

with \( M = M_N M_{N-1} \cdots M_2 M_1 \) ray transfer matrix for the optical system.
The thick lens and thin lens matrices

Goal: Applying the results for a thick lens

Let \( R \) represent a reflection matrix and \( T \) represent translation

\[
M = R_2 T R_1
\]

the ray-transfer matrix for a thick lens can be written as:

\[
M = \begin{bmatrix}
1 & 0 \\
\frac{n_L - n'}{n R_2} & \frac{n_L}{n'}
\end{bmatrix}\begin{bmatrix}
1 & t \\
0 & 1
\end{bmatrix}\begin{bmatrix}
1 & 0 \\
\frac{n - n_L}{n L R_1} & \frac{n}{n_L}
\end{bmatrix}
\]

For a thin lens \( t \to 0 \) in one environment \( (n = n') \) the ray-transfer matrix becomes

\[
M = \begin{bmatrix}
1 & 0 \\
\frac{n_L - n}{n R_2} & \frac{n_L}{n}
\end{bmatrix}\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}\begin{bmatrix}
1 & 0 \\
\frac{n - n_L}{n L R_1} & \frac{n}{n_L}
\end{bmatrix} = \left[ \frac{n_L - n}{n} \left( \frac{1}{R_2} - \frac{1}{R_1} \right) 1 \right]
\]

We express the lower left hand element in terms of the focal length:

\[
\frac{1}{f} = \frac{n_L - n}{n} \left( \frac{1}{R_2} - \frac{1}{R_1} \right)
\]

the lensmaker's formula

\[
M = \begin{bmatrix}
1 & 0 \\
-\frac{1}{f} & 1
\end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}
\]

the ray-transfer matrix for a thin lens also known as the ABCD matrix.
### Table 18-1: Summary of Some Simple Ray-Transfer Matrices

- **Translation matrix:**
  \[
  M = \begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix} = \mathcal{X}
  \]

- **Refraction matrix, spherical interface:**
  \[
  M = \begin{bmatrix} 1 \\ 0 \\ n-n' \\ \frac{n}{Rn'} \end{bmatrix} = \mathcal{R}
  \]

- **Refraction matrix, plane interface:**
  \[
  M = \begin{bmatrix} 1 & 0 \\ 0 & \frac{n}{n'} \end{bmatrix}
  \]

- **Thin-lens matrix:**
  \[
  M = \begin{bmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{bmatrix}
  \]
  \[
  \frac{1}{f} = \frac{n'-n}{n} \left( \frac{1}{R_1} - \frac{1}{R_2} \right)
  \]

- **Spherical mirror matrix:**
  \[
  M = \begin{bmatrix} 1 & 0 \\ \frac{2}{R} & 1 \end{bmatrix}
  \]

(+R): convex  
(-R): concave
Significance of system matrix elements

\[
\begin{bmatrix}
  y_f \\
  \alpha_f
\end{bmatrix} =
\begin{bmatrix}
  A & B \\
  C & D
\end{bmatrix}
\begin{bmatrix}
  y_0 \\
  \alpha_0
\end{bmatrix}
\]

a) If \( D = 0 \) \( \rightarrow \) \( \alpha_f = Cy_0 \) independent of \( \alpha_0 \)
All the rays leaving the input plane will have the same angle at the output plane.
Input plane is on the first focal plane.
b) If \( A = 0 \) \( \rightarrow \) \( y_f = B\alpha_0 \)
means \( y_f \) is independent of \( y_0 \) that means all the rays departing input plane have the same height at the output plane. This means output plane is the second focal plane.
c) If \( B = 0 \) \( \rightarrow \) \( y_f = Ay_0 \) All the points leaving the input plane at height \( y_0 \) will arrive the output plane at height \( y_f \) output plane is image of the input plane.
\( A=y_f/y_0 \) corresponds to linear magnification.
d) If \( C = 0 \) \( \rightarrow \) \( \alpha_f = D\alpha_0 \) independent of \( y_0 \)
Input rays of all in one direction will produce output rays all in another direction.
This is called (thelescopic system).
Example 18.3
Location of cardinal points for an optical system

Since the system ray-transfer matrix explains the optical properties of an optical system we expect a relationship between the system matrix and location of the cardinal points.

Input and output planes define limits of an optical system.

We define distances locating six cardinal planes with respect to the input and output planes.

- $F_1$ and $F_2$ are at $f_1$ and $f_2$ from the principal points at $H_1$ and $H_2$.
- $F_1^{}$ and $F_2^{}$ are at $p$ and $q$ from the reference input and output planes.
- $r$ and $s$ are distances of the reference input and output planes from the principal points at $H_1$ and $H_2$.
- $v$ and $w$ are distances of the reference input and output planes from the nodal points at $N_1$ and $N_2$.

Sign convention:

- (+) distance measured to the right of a reference plane.
- (-) distance measured to the left of a reference plane.

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Location of cardinal points

Input ray \((y_0, \alpha_0)\) and output ray \((y_f, 0)\) from figure (a)

\[
y_f = Ay_0 + B\alpha_0
\]
\[
0 = Cy_0 + D\alpha_0
\]

\[
y_0 = -\left(\frac{D}{C}\right)\alpha_0
\]

For small angles \(\alpha_0 = \frac{y_0}{-p} \rightarrow p = -\frac{y_0}{\alpha_0} = \frac{D}{C}
\]

\(p\) is negative that means it is to the left of the input reference plane.

\[
\alpha_0 = \frac{y_f}{-f_1}
\]
\[
f_1 = \frac{-y_f}{\alpha_0} = \frac{-(Ay_0 + B\alpha_0)}{\alpha_0} = \frac{AD}{C} - B
\]

\[
f_1 = \frac{AD - BC}{C} = \frac{Det(M)}{C} = \left(\frac{n_0}{n_f}\right)\frac{1}{C} = f_1
\]

We used: \(Det(M) = AD - BC = \frac{n_0}{n_f}\)

\[
r = p - f_1 = \frac{1}{C}\left(D - \frac{n_0}{n_f}\right) = r
\]
Location of cardinal points

Using figure (b) we can find $q, f_2, s$

\[
q = -\frac{A}{C}; \quad s = \frac{1 - A}{C}; \quad f_2 = q - s = -\frac{1}{C}
\]

Using figure (c) we can find $v, w$

\[
\alpha_0 = \alpha_f = \alpha = -\frac{y_0}{v}
\]

Notice $y_0$ is negative i.e. below the optical axis

\[
\alpha = Cy_0 + D\alpha \rightarrow \frac{y_0}{\alpha} = \frac{1 - D}{C} = -v
\]

\[
v = \frac{D - 1}{C} \quad \text{and} \quad w = \frac{\left(n_0 / n_f\right) - A}{C}
\]
1) When initial and final material have the same index of refraction then \( r = v \) and \( s = w \), i.e., principal points and nodal points coincide.

2) When initial and final material have the same index of refraction then first and second focal lengths are equal \( f_1 = f_2 \).

3) The separation of the principal points is the same as separation of the nodal points or \( r - s = v - w \).
Examples: tow thin lenses in air separated by a distance L
Examples: tow thin lenses in air separated by a distance L

Focal lengths of the lenses $f_A, f_B$,

Assume the input and output reference planes are located on the lenses.

The system transfer matrix includes two thin-lens matrices and a translation matrix.

$$M = \begin{bmatrix} 1 & 0 \\ \frac{-1}{f_B} & 1 \end{bmatrix} \begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \frac{-1}{f_B} & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{f_B} \left( \frac{1}{f_A} - 1 \right) & 1 - \frac{L}{f_B} \\ 1 & 1 - \frac{L}{f_A} \end{bmatrix}$$

First focal length of the system: $f_1 = \frac{1}{C}$ and the second focal length of the system: $f_2 = -\frac{1}{C} = f_{eq}$

$$\frac{1}{f_{eq}} = \frac{1}{f_A} + \frac{1}{f_B} - \frac{L}{f_A f_B}$$

The first principal point and nodal point: $r = v = \left( \frac{f_{eq}}{f_B} \right) L$

The second principal point and nodal point: $s = w = \left( \frac{f_{eq}}{f_A} \right) L$
Example
Ray tracing

Limiting analysis of optical systems to paraxial rays is an over simplification of the problem and ignores effect of aberrations.

Ray tracing is following the actual path of each ray through the system using laws of reflection and refraction. Traditionally it is done by hand and graphically but today it is all computerized.

We introduce a ray-tracing technique that is often limited to meridional rays.

Meridional rays are the rays that pass through the optical axis of the system. Meridional rays tend to stay in the meridional planes as the laws of refraction/reflection require them.

This limits our treatment to a 2-dimensional space.

Skew rays are the ones that contribute to the image and do not pass the optical axis. Analysis of the skew rays require a 3-dimensional treatment.

Understanding aberrations require analysis of the non-paraxial rays and skew rays.

Design of the complicated lens systems require knowledge and experience with ray-tracing techniques and optimizing performance of the system by changing system parameters and arriving at a perfect performance.
Ray tracing
Goal: follow a meridional ray through a single spherical refracting surface.

\( n, n' \): Indexes of refraction; \( R \) radius of curvature; \( A \) origin of the ray.
\( \alpha, \alpha' \): angle with optical axis before and after refraction.

\( O \) point of intersection with the optical axis; \( P \) with the refracting surface; \( I \) with the optical axis after refraction.
\( I \) & \( O \) are conjugate points with distances \( s, s' \) from the vertices.

\( Q \): perpendicular distance from the vertex, \( V \), to the incident ray.
\( \theta, \theta' \): angles of incidence and refraction.

Sign convention: distances to the left of vertex \( - \) and to the right of \( V \) are \( + \) above the optical axis \( + \) and below are \( - \).

From left to right, the angles have the same sign as the slopes.

Input parameters for each ray: \( h \): elevation, \( \alpha \): angle, and \( D \): distance from the vertex parallel to the optical axis.

From figure we write:

\[
\sin \theta = \frac{R \sin \alpha + Q}{R} \quad \sin \theta' = \frac{R \sin \alpha + Q'}{R}
\]

\[
\theta = \sin^{-1} \left( \frac{Q}{R} + \sin \alpha \right)
\]

At \( P \):

\[
\sin \theta = n \sin \theta' \rightarrow \theta' = \sin^{-1} \left( \frac{n \sin \theta}{n'} \right)
\]

\[
\sin \theta - \alpha = \theta' - \alpha' \rightarrow \alpha' = \theta' - \theta + \alpha
\]
**Ray tracing**

Q': perpendicular distance from the vertex, V, to the refracted ray.

\[
\begin{align*}
\text{In } \triangle CMV & \quad \sin(-\alpha') = \frac{a'}{R} \\
\text{In } \triangle PLC & \quad \sin(\theta') = \frac{Q' - a'}{R} \\
\text{In } \triangle ITV & \quad \sin(-\alpha') = \frac{Q'}{s'} 
\end{align*}
\]

\[Q' = R \left( \sin \theta' - \sin \alpha' \right)\]

Now we have the new values for the refracted ray \(\alpha', Q', s'\) which prepare us for the next refraction in the sequence. But before that we need to calculate effect of the transfer by \(t\) in the material with index \(n'\).

\[
\begin{align*}
\text{In } \triangle V_2MV_1 & \quad \sin(-\alpha_2) = \frac{V_1M}{t} = \frac{Q'_1 - Q_2}{t} \\
\alpha_2 = \alpha'
\end{align*}
\]

And \(\alpha_2 = \alpha'\)

We need to modify the equations for the special cases:

1) Incident ray is parallel to the optical axis.
2) Surface is plane with an infinite radius of curvature.
<table>
<thead>
<tr>
<th>General case</th>
<th>Ray parallel to axis: $\alpha = 0$</th>
<th>Plane surface: $R \gg \infty$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s = D - \frac{h}{\tan \alpha}$</td>
<td>$-$</td>
<td>$s = D - \frac{h}{\tan \alpha}$</td>
</tr>
<tr>
<td>$Q = -s \sin \alpha$</td>
<td>$Q = h$</td>
<td>$Q = -s \sin \alpha$</td>
</tr>
<tr>
<td>$\theta = \sin^{-1}\left(\frac{Q}{R} + \sin \alpha\right)$</td>
<td>$\theta = \sin^{-1}\left(\frac{Q}{R} + \sin \alpha\right)$</td>
<td>$-$</td>
</tr>
<tr>
<td>$\theta' = \sin^{-1}\left(\frac{n \sin \theta}{n'}\right)$</td>
<td>$\theta' = \sin^{-1}\left(\frac{n \sin \theta}{n'}\right)$</td>
<td>$-$</td>
</tr>
<tr>
<td>$\alpha' = \theta' - \theta + \alpha$</td>
<td>$\alpha' = \theta' - \theta + \alpha$</td>
<td>$\alpha' = \sin^{-1}\left(\frac{n}{n'} \sin \alpha\right)$</td>
</tr>
<tr>
<td>$Q' = R(\sin \theta' - \sin \alpha')$</td>
<td>$Q' = R(\sin \theta' - \sin \alpha')$</td>
<td>$Q' = \frac{Q \cos \alpha'}{\cos \alpha}$</td>
</tr>
<tr>
<td>$s' = \frac{-Q'}{\sin \alpha'}$</td>
<td>$s' = \frac{-Q'}{\sin \alpha'}$</td>
<td>$s' = \frac{-Q'}{\sin \alpha'}$</td>
</tr>
</tbody>
</table>

*Transfer: Input: $t$*

$Q = Q' + t \sin \alpha'$

$\alpha = \alpha'$

$n = n'$

*Input: new $n'$, $R$*

*Return: to calculate $\theta$*
Example ray tracing

Do a ray trace for two rays through a rapid landscape photographic lens of three elements. The parallel rays enter the lens from a distant object at altitudes of 1 and 5 mm above the optical axis. The lens specifications are:

<table>
<thead>
<tr>
<th>First surface</th>
<th>Input</th>
<th>Results ray at h=1</th>
<th>Results ray at h=5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n = 1, n' = 1.521$</td>
<td>$Q = 1$</td>
<td>$Q = 5$</td>
<td></td>
</tr>
<tr>
<td>$\alpha = 0$</td>
<td>$\alpha' = 0.1625^0$</td>
<td>$\alpha' = 0.8128^0$</td>
<td></td>
</tr>
<tr>
<td>$h = 1$ or $5$</td>
<td>$s' = -352.66$</td>
<td>$s' = -352.53$</td>
<td></td>
</tr>
<tr>
<td>$R = -120.8$</td>
<td>$Q' = 1.0000$</td>
<td>$Q' = 5.0010$</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Second surface</th>
<th>Input</th>
<th>Results ray at h=1</th>
<th>Results ray at h=5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t = 6$</td>
<td>$Q = 1.0170$</td>
<td>$Q = 5.0861$</td>
<td></td>
</tr>
<tr>
<td>$n = 1.581$</td>
<td>$\alpha' = 0.2202^0$</td>
<td>$\alpha' = 1.1041^0$</td>
<td></td>
</tr>
<tr>
<td>$R = -34.6$</td>
<td>$s' = -264.59$</td>
<td>$s' = -264.03$</td>
<td></td>
</tr>
<tr>
<td>$Q' = 1.0170$</td>
<td>$Q' = 5.0876$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Third surface</th>
<th>Input</th>
<th>Results ray at h=1</th>
<th>Results ray at h=5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t = 2$</td>
<td>$Q = 1.0247$</td>
<td>$Q = 5.1261$</td>
<td></td>
</tr>
<tr>
<td>$n = 1.514$</td>
<td>$\alpha' = 0.2030^0$</td>
<td>$\alpha' = 1.0178^0$</td>
<td></td>
</tr>
<tr>
<td>$R = -96.2$</td>
<td>$s' = -289.26$</td>
<td>$s' = -288.58$</td>
<td></td>
</tr>
<tr>
<td>$Q' = 1.0247$</td>
<td>$Q' = 5.1260$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Final surface</th>
<th>Input</th>
<th>Results ray at h=1</th>
<th>Results ray at h=5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t = 3$</td>
<td>$Q = 1.0353$</td>
<td>$Q = 5.1793$</td>
<td></td>
</tr>
<tr>
<td>$n = 1.581$</td>
<td>$\alpha' = -0.2883^0$</td>
<td>$\alpha' = -1.4520^0$</td>
<td></td>
</tr>
<tr>
<td>$R = -51.2$</td>
<td>$</td>
<td>s' = -205.72</td>
<td>$</td>
</tr>
<tr>
<td>$Q' = 1.0353$</td>
<td>$Q' = 5.1672$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

There is no common focus $\Delta s' = 1.8\,\text{mm}$