Chapter 5
Superposition of Waves

Lecture Notes for Modern Optics based on
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Superposition of Waves

Studying combined effects of two or more harmonic waves.

Superposition of waves with different amplitudes and phases.

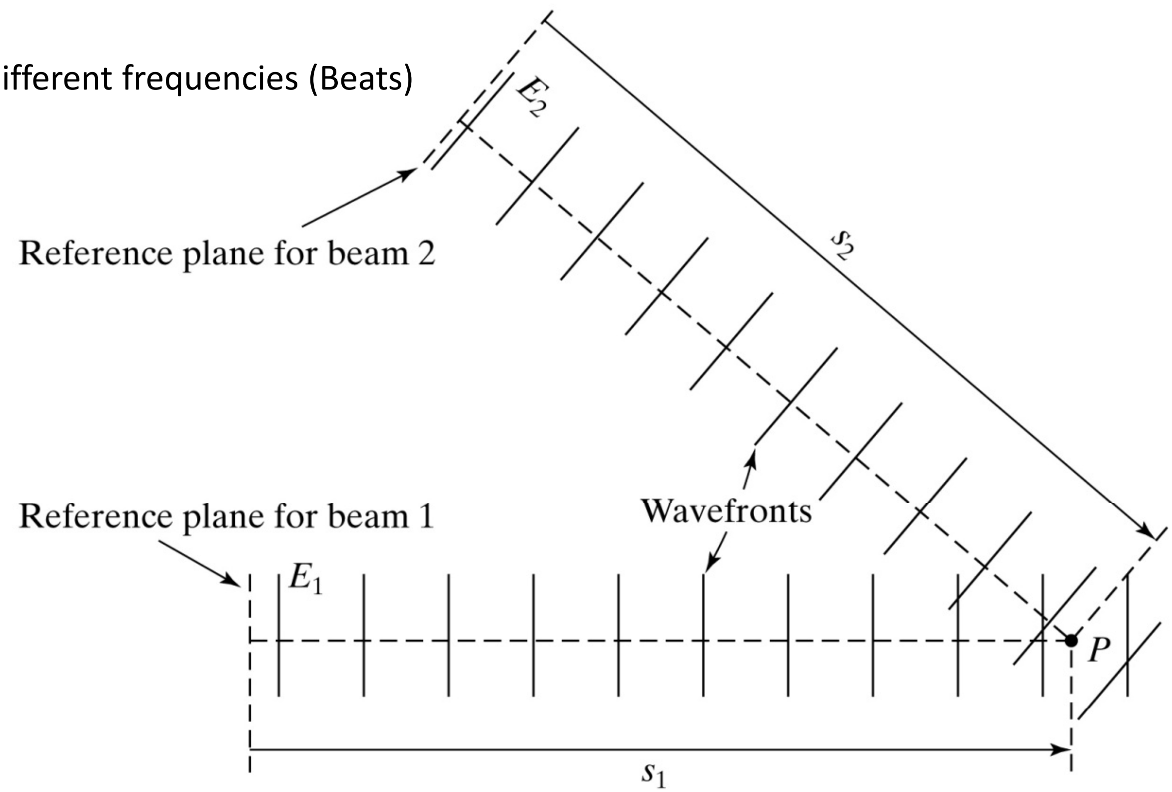
Irradiance attainable from randomly phased and coherent harmonic waves.

Superposition of waves with different frequencies.

Standing waves.

Superposition of the waves with slightly different frequencies (Beats)

Group velocity and phase velocity



Superposition Principle

What is the net displacement if two independent displacements coexist in a point of space. Acording to the superposition principl the net is sum of the individual displacements. $\psi = \psi_1 + \psi_2$. To test if this is the case we need to prove that $\psi = \alpha\psi_1 + \beta\psi_2$ is a solution of the wave equation.

$$\nabla^2\psi = \frac{1}{\mathrm{V}^2}\frac{\partial^2\psi}{\partial t^2}$$

Superposition of electromagnetic waves

is expresses in terms of **E** and **B** field vectors.

$$\mathbf{E} = \mathbf{E_1} + \mathbf{E_2} \qquad \mathbf{B} = \mathbf{B_1} + \mathbf{B_2}$$

In general oreintation of the fields is important

but for now we treat the fields as scaler for the

case that they are parallel or nearly parallel.

Superposition of waves same frequency

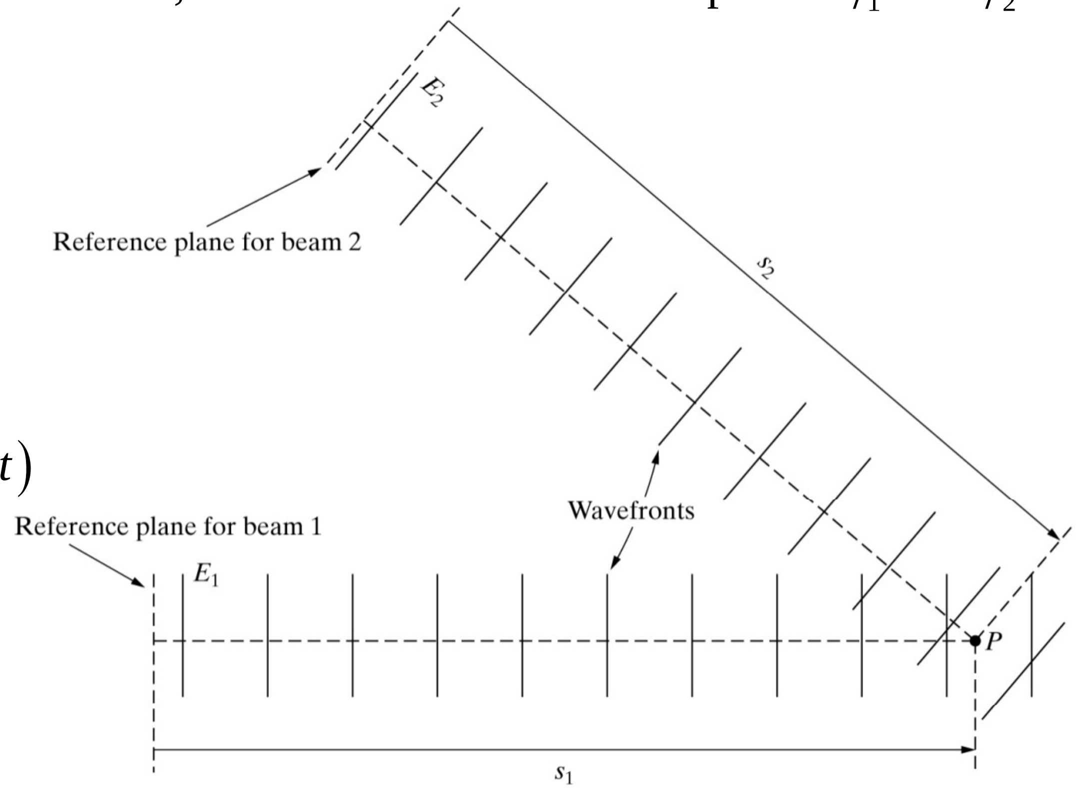
Two harmonic plane waves of the same frequency arrive at point P in space.

$$E_1 = E_{01}\cos\left(ks_1 - \omega t + \phi_1\right) = E_{01}\cos\left(\alpha_1 - \omega t\right) \text{ where } \alpha_1 = ks_1 + \phi_1$$

$$E_1 = E_{02}\cos\left(ks_2 - \omega t + \phi_2\right) = E_{02}\cos\left(\alpha_2 - \omega t\right) \text{ where } \alpha_2 = ks_2 + \phi_2$$

s_2 and s_2 are the directed distances along the propagation direction of each wave from the reference plane. On the reference planes at $t = 0$, the individual waves have phases ϕ_1 and ϕ_2 .

$$\underbrace{\alpha_2 - \alpha_1}_{\text{Phase difference of the waves arriving at P}} = k\underbrace{\left(s_2 - s_1\right)}_{\text{Optical path difference}} + \underbrace{\left(\phi_2 - \phi_1\right)}_{\text{Initial phase difference}}$$



The resulting electric field E_R is:

$$E_R = E_1 + E_2$$

$$E_R = E_{01}\cos\left(\alpha_1 - \omega t\right) + E_{02}\cos\left(\alpha_2 - \omega t\right)$$

Three cases are recognized.

Interference

<u>Constructive interference:</u>

The individual waves that are superimposed are "in step" or "in phase". The resulting wave is also "in step" with the original waves. In this case E_R is the sum of the amplitudes. Two waves of the same frequence interfere constructively if their phase difference is: $\Delta\phi = \alpha_2 - \alpha_1 = m(2\pi)$

$$E_R = E_{01}\cos(\alpha_1 - \omega t) + E_{02}\cos(\alpha_1 + 2m\pi - \omega t)$$

$$\boxed{E_{R\_constructive} = (E_{01} + E_{02})\cos(\alpha_1 - \omega t)}$$

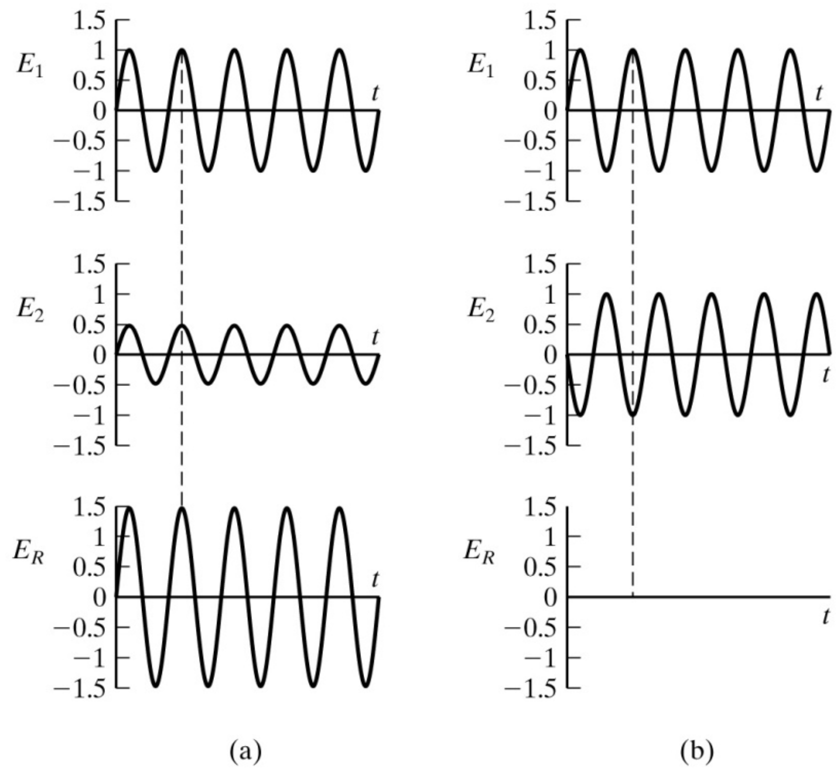
<u>Destructive interference:</u>

The individual waves that are superimposed are "out of step" or "out of phase". In this case E_R is the difference of the amplitudes. Two waves of the same frequence interfere destructively if their phase difference is: $\Delta\phi = \alpha_2 - \alpha_1 = (2m+1)\pi$

$$E_R = E_{01}\cos(\alpha_1 - \omega t) + E_{02}\cos(\alpha_1 + (2m+1)\pi - \omega t)$$

$$E_R = E_{01}\cos(\alpha_1 - \omega t) - E_{02}\cos(\alpha_1 - \omega t)$$

$$\boxed{E_{R\_destructive} = (E_{01} - E_{02})\cos(\alpha_1 - \omega t)}$$



Superposition General Case

Goal: find amplitude and phase of the two waves with the same frequency arriving at a point P.

It is simpler to treat this case with complex form of the fields.

$$\mathrm{E_R} = \mathrm{Re}\left(E_{01}e^{i(\alpha_1-\omega t)} + E_{02}e^{i(\alpha_2-\omega t)}\right) = \mathrm{Re}\left(e^{-i\omega t}\left(E_{01}e^{i\alpha_1} + E_{02}e^{i\alpha_2}\right)\right)$$

$$E_0 e^{i\alpha} = \left(E_{01}e^{i\alpha_1} + E_{02}e^{i\alpha_2}\right)$$

$$\mathrm{E_R} = \mathrm{Re}\left(E_0 e^{i(\alpha-\omega t)}\right) \rightarrow E_R = E_0\cos\left(\alpha-\omega t\right)$$

E_0 is the amplitude and α is the phase of the resulting wave at $t=0$. Using the vector form for the complex numbers (phasors) we find the E_0 and α.

Components of the fields on real and imaginary axis:

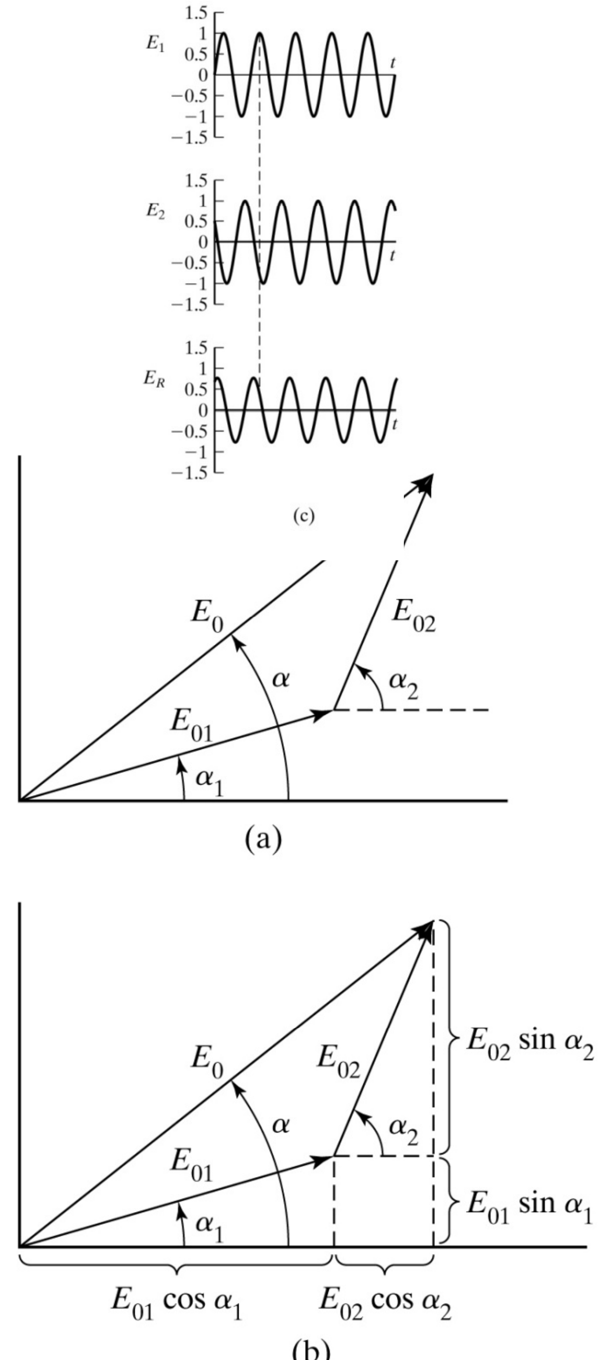
$$E_0\cos\alpha = E_{01}\cos\alpha_1 + E_{02}\cos\alpha_2$$

$$E_0\sin\alpha = E_{01}\sin\alpha_1 + E_{02}\sin\alpha_2$$

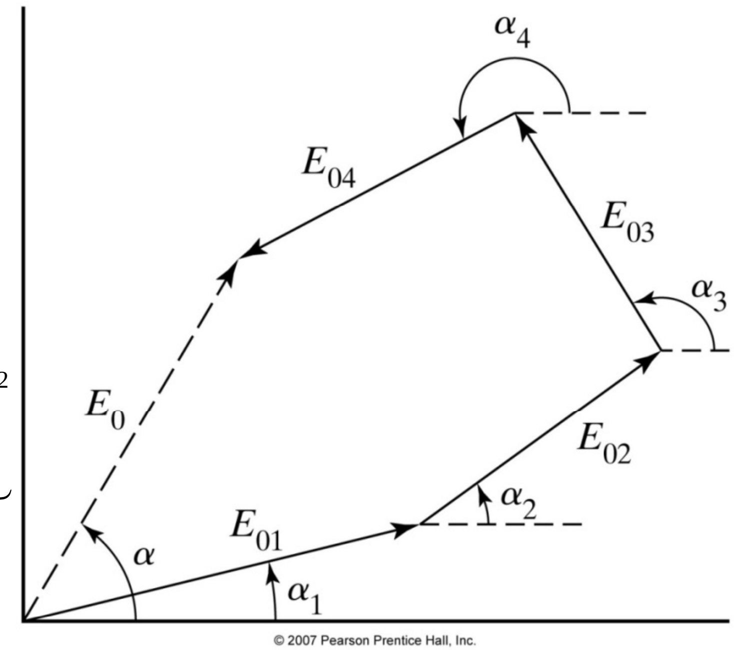
$E_R = E_0\cos\left(\alpha-\omega t\right)$ where

$$E_0^2 = E_{01}^2 + E_{02}^2 + 2E_{01}E_{02}\cos\left(\alpha_2-\alpha_1\right)$$

$$\tan\alpha = \frac{E_{01}\sin\alpha_1 + E_{02}\sin\alpha_2}{E_{01}\cos\alpha_1 + E_{02}\cos\alpha_2}$$



Superposition of many waves



Superposition of N harmonic waves with identical frequency.

The resulting electric field amplitude and phase are given by:

$$\tan\alpha = \frac{\sum_{i=1}^{N} E_{0i}\sin\alpha_i}{\sum_{i=1}^{N} E_{0i}\cos\alpha_i} \quad \text{and} \quad \underbrace{E_0^2 = \left(\sum_{i=1}^{N} E_{0i}\sin\alpha_i\right)^2 + \left(\sum_{i=1}^{N} E_{0i}\cos\alpha_i\right)^2}_{\text{The Pythagorean theorem}}$$

$$\left(\sum_{i=1}^{N} E_{0i}\sin\alpha_i\right)^2 = \sum_{i=1}^{N} E_{0i}^2 \sin^2\alpha_i + 2\sum_{j>i}^{N}\sum_{i=1}^{N} E_{0i}E_{0j}\sin\alpha_i\sin\alpha_j$$

$$\left(\sum_{i=1}^{N} E_{0i}\cos\alpha_i\right)^2 = \underbrace{\sum_{i=1}^{N} E_{0i}^2 \cos^2\alpha_i}_{\substack{\text{Sum of the squares of}\\ \text{the individual terms}}} + \underbrace{2\sum_{j>i}^{N}\sum_{i=1}^{N} E_{0i}E_{0j}\cos\alpha_i\cos\alpha_j}_{\substack{\text{Sum of the cross products}\\ \text{excluding the self-products}}}$$

$$E_0^2 = \sum_{i=1}^{N} E_{0i}^2\left(\sin^2\alpha_i + \cos^2\alpha_i\right) + 2\sum_{j>i}^{N}\sum_{i=1}^{N} E_{0i}E_{0j}\left(\cos\alpha_i\cos\alpha_j + \sin\alpha_i\sin\alpha_j\right)$$

Sum of N harmonic waves with identical frequency is a harmonic wave of the same frequency but

amplitude of $\boxed{E_0^2 = \sum_{i=1}^{N} E_{0i}^2 + 2\sum_{j>i}^{N}\sum_{i=1}^{N} E_{0i}E_{0j}\left(\cos\left(\alpha_j - \alpha_j\right)\right)}$ and phase of $\boxed{\tan\alpha = \frac{\sum_{i=1}^{N} E_{0i}\sin\alpha_i}{\sum_{i=1}^{N} E_{0i}\cos\alpha_i}}$

Random and coherent sources

Two important cases of superposition:

$$E_0^2 = \sum_{i=1}^{N} E_{0i}^2 + 2\sum_{j>i}^{N}\sum_{i=1}^{N} E_{0i}E_{0j}\left(\cos\left(\alpha_j - \alpha_i\right)\right)$$

Two important cases of superposition:

1) <u>Randomly phased sources of equal amplitudes</u> when N is a large number.

For this case the phase differences $\left(\alpha_j - \alpha_i\right)$ are random so $\lim\limits_{N\to\text{large}} \sum_{j>i}^{N}\sum_{i=1}^{N} E_{0i}E_{0j}\left(\cos\left(\alpha_j - \alpha_i\right)\right) \to 0$

$$\boxed{E_0^2 = \underbrace{\sum_{i=1}^{N} E_{0i}^2}_{\substack{\text{Sum of the square}\\ \text{of the amplitudes}}} = \underbrace{NE_{01}^2}_{\substack{\text{Irradiance of N non-coherent}\\ \text{sources is N times irradiance of}\\ \text{the individual source}}}}$$

2) N <u>coherent</u> $\Big($sources have <u>constant phase relationship</u>$\Big)$ sources of the same type and <u>in phase</u> $\left(\alpha_j - \alpha_i = 0\right)$.

$$E_0^2 = \sum_{i=1}^{N} E_{0i}^2 + 2\sum_{j>i}^{N}\sum_{i=1}^{N} E_{0i}E_{0j} \to \boxed{E_0^2 = \underbrace{\left(\sum_{i=1}^{N} E_{0i}\right)^2}_{\substack{\text{Square of the sum}\\ \text{of the amplitudes}}} = \left(\mathrm{NE}_{01}\right)^2 = \underbrace{N^2E_{01}^2}_{\substack{\text{Irradiance of N coherent}\\ \text{sources is N}^2\text{ times irradiance}\\ \text{of the individual source}}}}$$

Standing waves

A special case of superposition is when waves exist in both forward and bacward directions in a medium.

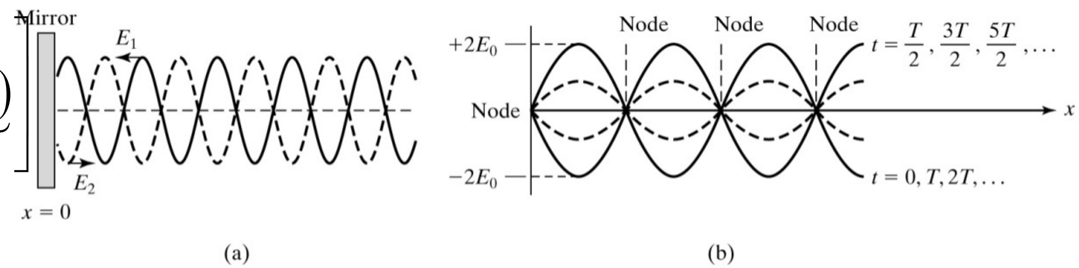
Example of such a case is when waves are reflected by a surface.

If there is no loss of enery due to reflection or transmission, the amplitude stays the same.

$E_1 = E_0 \sin(\omega t + kx)$ the wave in $-x$ directon

$E_2 = E_0 \sin(\omega t - kx - \phi_R)$ the wave in $+x$ directon with a possible phase shift ϕ_R upon reflection.

$$E_R = E_1 + E_2 = E_0\left[\underbrace{\sin(\omega t + kx)}_{\beta_+} + \underbrace{\sin(\omega t - kx - \phi_R)}_{\beta_-}\right]$$



$$\sin\beta_+ + \sin\beta_- = 2\sin\left(\frac{\beta_+ + \beta_-}{2}\right)\cos\left(\frac{\beta_+ - \beta_-}{2}\right)$$

$$E_R = 2E_0 \cos(kx + \phi_R/2)\sin(\omega t - \phi_R/2)$$

In the case of reflection from a plane conducting mirror $\phi_R = \pi$ and E_R is:

$$E_R = 2E_0\cos(kx + \pi/2)\sin(\omega t - \pi/2) = \underbrace{\left(2E_0 \sin(kx)\right)}_{A(x)\text{ space dependnt amplitude}} \cos(\omega t) \to \boxed{E_R = A(x)\cos(\omega t)}$$

$E_R = 0$ at any time if $kx = m\pi,\ m = 0, \pm 1, \pm 2, \ldots\ \ A(x) = \sin(kx) = 0$

$x = m\lambda/2 = 0, \lambda/2, \lambda, 3\lambda/2, \ldots$ nodes of standing waves separated by $\lambda/2$

$E_R = E_{\max}$ at any time if $\omega t = m\pi = 2\pi\nu t = (2\pi/T)t,\ m = 0, \pm 1, \pm 2, \ldots$

So for $t = m(T/2) = 0, T/2, T, 3T/2, \ldots$ we have $E_R = E_{\max}$

And for $t = T/4, 3T/4, \ldots$ $\cos\omega t = 0$ and thus $E_R = 0$

Standing waves in a laser cavity

A laser cavity ois composed of two reflecting mirrors separatd by a distance d.

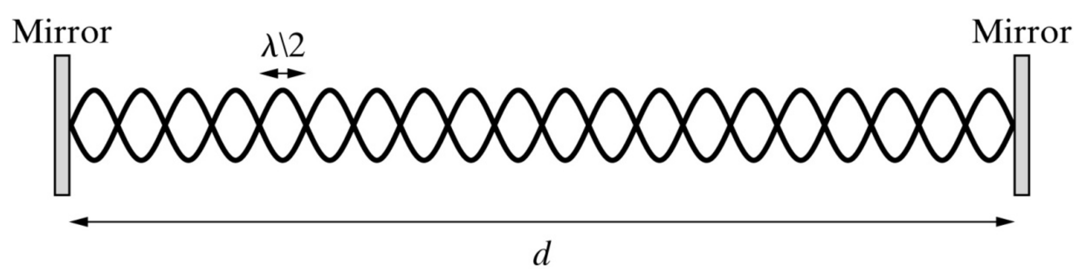
Light generated by the laser material is reflected back and forth so it is acase of superposition of waves with the same frequency moving in opposite directions.

Boundary conditions dictate that E=0 at the mirrors equivalent to a node. Therefore the cavity will only support the wavelengths that can generate nodes at the mirrors. This translates to:

$$d = m\frac{\lambda_m}{2} \quad \text{where } m = 1, 2, 3, \ldots \rightarrow \lambda_m = \frac{2d}{m} \rightarrow \nu_m = \frac{c}{\lambda_m} = m\frac{c}{2d}$$

λ_ms are called the standing wave normal modes of the cavity like the modes of a string.

Output of a laser will consists of those frequencies that the gain medium can produce and are part of the normal modes of the cavity.



Beat phenomenon

Superposition of the waves with the same or comparable amplitudes but different frequencies. Different frequencies means different wavelengths that leads to different speed for each frequency in dispersive media. Here we are only considering the case of non-dispersive media. Note both $\omega=2\pi\nu$, k=$2\pi\lambda$ are different for these waves:

$$\left.\begin{aligned} E_1 &= E_0\cos\left(k_1 x-\omega_1 t\right)\\ E_2 &= E_0\cos\left(k_2 x-\omega_2 t\right)\end{aligned}\right\} E_R = E_1+E_2 = E_0\left[\cos\left(k_1 x-\omega_1 t\right)+\cos\left(k_2 x-\omega_2 t\right)\right]$$

$$\cos\alpha+\cos\beta = 2\cos\frac{1}{2}(\alpha+\beta)\cos\frac{1}{2}(\alpha-\beta)$$

$$\mathrm{E_R} = 2E_0\cos\left[\underbrace{\frac{k_1+k_2}{2}}_{k_p}x-\underbrace{\frac{\omega_1+\omega_2}{2}}_{\omega_p}t\right]\cos\left[\underbrace{\frac{k_1-k_2}{2}}_{k_g}x-\underbrace{\frac{\omega_1-\omega_2}{2}}_{\omega_g}t\right]$$

$$\mathrm{E_R} = 2E_0\underbrace{\cos\left(k_p x-w_p t\right)}_{\substack{\text{cosine wave with}\\ \text{average frequency \&}\\ \text{propagation constant}}}\ \underbrace{\cos\left(k_g x-w_g t\right)}_{\substack{\text{cosine wave with a smaller}\\ \text{(difference) frequency \&}\\ \text{propagation constant}}}$$

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For $\omega_p >> \omega_g$ the resulting wave is shown.

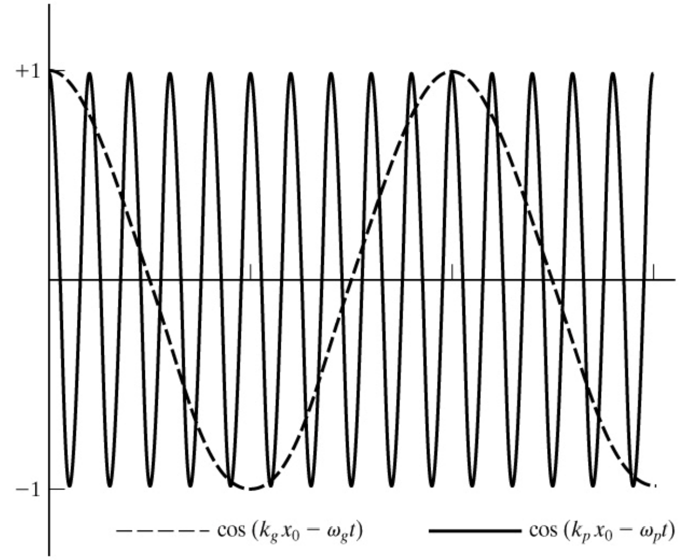
A high frequency wave that its amplitude is modulated by a low frequency cos wave.

The low frequency wave acts as the envelope for the amplitude of the high frequency wave.

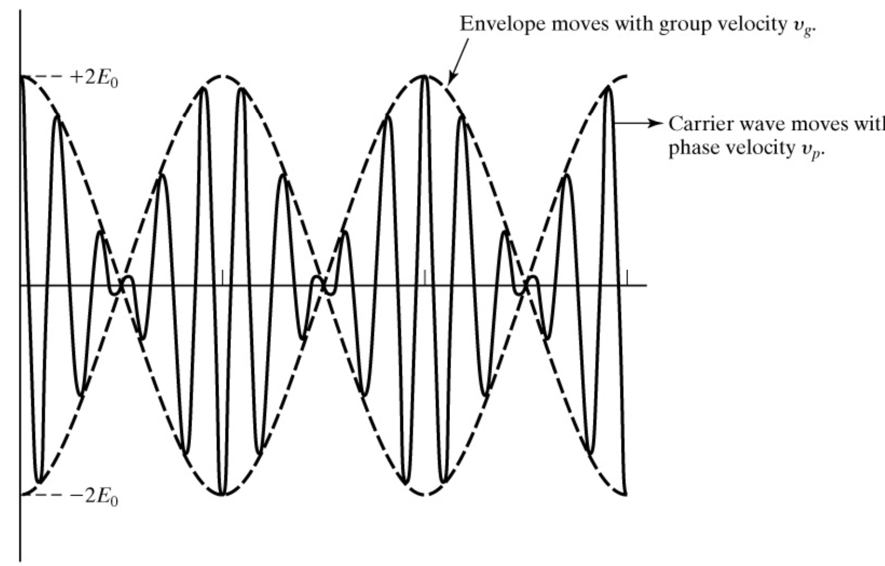
The energy delivered by such a wave has the beat frequency:

$$\boxed{\omega_{\text{beat}} = 2\omega_g = 2\left(\frac{\omega_1 - \omega_2}{2}\right) = \omega_1 - \omega_2}$$

This phenomonon is used to measure frequency differences, tuning the musical instruments.



(a)



(b)

Phase and group velocity

Any pilse of light can be reconstructed from superposition of harmonic waves with different frequencies. The shorter the pulse (in time) the larger number of frequencies required to buid it. Waves with different frequencies travel with different speeds in the medium (dispersion). Phase velocity: is the velocity of the harmonic waves that costitute the light signal. Group velocity: is the velocity at which the position of the maximal constructive interference propagate.

For the example of the beats $E_R = 2E_0 \underbrace{\cos\left(k_p x - w_p t\right)}_{\text{The harmonic wave}} \quad \underbrace{\cos\left(k_g x - w_g t\right)}_{\text{The envelope result of the interference}}$

$$k_p = \frac{k_1 + k_2}{2} \sim k,\ \omega_p = \frac{\omega_1 + \omega_2}{2} \sim \omega,\ k_g = \frac{k_1 - k_2}{2} \sim \frac{dk}{2},\ \omega_g = \frac{\omega_1 - \omega_2}{2} \sim \frac{d\omega}{2}$$

$$\begin{cases} kx - \omega t = 0 \rightarrow \boxed{V_p = \frac{\omega}{k}} \leftarrow \text{Velocity of the harmonic wave} \\ xdk - td\omega = 0 \rightarrow \boxed{V_g = \frac{d\omega}{dk}} \leftarrow \text{Velocity of the envelope} \end{cases}$$

$$V_g = \frac{d\omega}{dk} = \frac{d\left(V_p k\right)}{dk} \rightarrow \boxed{V_g = V_p + \frac{dV_p}{dk}}$$

In non-dispersive material or vacuum

$$\frac{dV_p}{dk} = 0 \rightarrow V_g = V_p = c$$

In dispersive materail:

$$V_p = \frac{c}{n(k)} \rightarrow V_g = V_p + \frac{d}{dk}\left(\frac{c}{n(k)}\right) = V_p - \frac{c}{n^2}\frac{dn}{dk}$$

$$\boxed{V_g = V_p\left(1 - \frac{k}{n}\frac{dn}{dk}\right) = V_p\left(1 + \frac{\lambda}{n}\frac{dn}{d\lambda}\right)}$$